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ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA AND OTHER RELATED VARIETIES.

JEAN-YVES CHARBONNEL AND MOUCHIRA ZAITER

ABSTRACT. In this note, one discusses about some varieties which are constructed analogously to the isospectral commuting varieties. These varieties are subvarieties of varieties having very simple desingularizations. For instance, this is the case of the nullcone of any cartesian power of a reductive Lie algebra and one proves that it has rational singularities. Moreover, as a byproduct of these investigations and the Ginzburg's results, one gets that the normalizations of the isospectral commuting variety and the commuting variety have rational singularities.

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1. INTRODUCTION.

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. The neutral element of G is denoted by $1_{\mathfrak{g}}$.

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1.1. **Notations.** • For V a vector space, its dual is denoted by V^* and the augmentation ideal of its symmetric algebra $\mathrm{S}(V)$ is denoted by $\mathrm{S}_+(V)$.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, let denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is bigger than 2. For Y a closed subset of an algebraic variety X, its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\mathcal{O}_X$ is its structural sheaf, $\Bbbk[X]$ is the algebra of regular functions on X and $\Bbbk(X)$ is the field of rational functions on X when X is irreducible. When X is smooth, the sheaf of regular differential forms of top degree on X is denoted by Ω_X.

• For X an algebraic variety and for $\mathcal{M}$ a sheaf on X, $\Gamma(V, \mathcal{M})$ is the space of local sections of $\mathcal{M}$ over the open subset V of X. For i a nonnegative integer, $\mathrm{H}^i(X, \mathcal{M})$ is the i-th group of cohomology of $\mathcal{M}$. For example, $\mathrm{H}^0(X, \mathcal{M}) = \Gamma(X, \mathcal{M})$.

Lemma 1.1. *Let X be an irreducible affine algebraic variety and let Y be a desingularization of X. Then $\mathrm{H}^0(Y, \mathcal{O}_Y)$ is the integral closure of $\Bbbk[X]$ in its fraction field.*

Proof. Let X_{n} be the normalization of X. According to [H77, Ch. II, Exercise 3.8], the desingularization morphism factorizes through X_{n} so that Y is a desingularization of X_{n}. So one can suppose $X = X_{\mathrm{n}}$. Then $\Bbbk[X]$ is a subalgebra of $\mathrm{H}^0(Y, \mathcal{O}_Y)$. Moreover, $\mathrm{H}^0(Y, \mathcal{O}_Y)$ is a subalgebra of $\Bbbk(X)$ since Y is a desingularization of X. According to [H77, Ch. II, Proposition 4.1], a morphism of affine varieties is separated. Then, according to [EGAII, Corollaire 5.4.3], $\mathrm{H}^0(Y, \mathcal{O}_Y)$ is a finite extension of $\Bbbk[X]$ since it is finitely generated and since the desingularization morphism is projective by definition, whence the lemma. □

• For K a group and for E a set with a group action of K, E^K is the set of invariant elements of E under K.

Lemma 1.2. *Let A be an algebra generated by the subalgebras A_1 and A_2. Let K be a group with a group action of K on A_2. Let suppose that the following conditions are verified:*

(1) *$A_1 \cap A_2$ is contained in A_2^K,*
(2) *A is a free A_2-module having a basis contained in A_1,*
(3) *A_1 is a free $A_1 \cap A_2$-module having the same basis.*

Then there exists a unique group action of K on the algebra A extending the action of K on A_2 and fixing all the elements of A_1. Moreover, if $A_1 \cap A_2 = A_2^K$ then $A^K = A_1$.

Proof. Let $m_l, l \in L$ be a basis of the A_2-module A, contained in A_1, and let M be the subspace of A generated by the m_l's so that the canonical morphisms

$$M \otimes_{\Bbbk} A_2 \longrightarrow A \qquad M \otimes_{\Bbbk} (A_1 \cap A_2) \longrightarrow A_1$$

are isomorphisms by Conditions (2) and (3). Hence there exists a unique group action of K on the space A fixing all the elements of M and extending the action of K on A_2. For (i, j) in L^2, let denote by $a_{i,j,k}$ the coordinate of $m_i m_j$ at m_k in the basis $m_l, l \in L$. According to Conditions (1) and (3), the $a_{i,j,k}$'s are invariant under K. Let a, a' be in A. Denoting by a_i and a'_i the coordinates of a and a' at m_i in the basis $m_l, l \in L$ respectively, for all g in K, one has

$$\begin{aligned} g.aa' &= g.(\textstyle\sum_{(i,j)\in L^2} m_i m_j a_i a'_j) \\ &= g.(\textstyle\sum_{k\in L} m_k(\sum_{(i,j)\in L^2} a_{i,j,k} a_i a'_j)) \\ &= \textstyle\sum_{k\in L} m_k(\sum_{(i,j)\in L^2} a_{i,j,k}(g.a_i)(g.a'_j)) \\ &= \textstyle\sum_{(i,j)\in L^2} m_i m_j (g.a_i)(g.a'_j) \\ &= (g.a)(g.a') \end{aligned}$$

so that the action of K is an action on the algebra A, fixing all element of A_1. Furthermore, a is in A^K if and only if the a_i's are in A_2^K since the m_l's are invariant under K. Hence $A^K = A_1$ if $A_1 \cap A_2 = A_2^K$. □

• For E a set and k a positive integer, E^k denotes its k-th cartesian power. If E is finite, its cardinality is denoted by $|E|$. If E is a vector space, for $x = (x_1, \dots, x_k)$ in E^k, P_x is the subspace of E generated by $x_1, \dots, x_k$. Moreover, there is a canonical action of $\mathrm{GL}_k(\Bbbk)$ in E^k given by:

$$(a_{i,j}, 1 \leq i, j \leq k).(x_1, \dots, x_k) := (\sum_{j=1}^{k} a_{i,j} x_j, i = 1, \dots, k)$$

In particular, the diagonal action of G in $\mathfrak{g}^k$ commutes with the action of $\mathrm{GL}_k(\Bbbk)$.

• For $\mathfrak{a}$ reductive Lie algebra, its rank is denoted by $\ell_{\mathfrak{a}}$ and the dimension of its Borel subalgebras is denoted by $\mathrm{b}_{\mathfrak{a}}$. In particular, $\dim \mathfrak{a} = 2\mathrm{b}_{\mathfrak{a}} - \ell_{\mathfrak{a}}$.

• If E is a subset of a vector space V, let denote by $\mathrm{span}(E)$ the vector subspace of V generated by E. The grassmanian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$. By definition, a *cone* of V is a subset of V invariant under the natural action of $\Bbbk^* := \Bbbk \setminus \{0\}$ and a *multicone* of V^k is a subset of V^k invariant under the natural action of $(\Bbbk^*)^k$ on V^k.

Lemma 1.3. *Let X be an open cone of V and let S be a closed multicone of $X \times V^{k-1}$. Denoting by S_1 the image of S by the first projection, $S_1 \times \{0\} = S \cap (X \times \{0\})$. In particular, S_1 is closed in X.*

Proof. For x in X, x is in S_1 if and only if for some $(v_2, \dots, v_k)$ in V^{k-1}, $(x, tv_2, \dots, tv_k)$ is in S for all t in $\Bbbk$ since S is a closed multicone of $X \times V^{k-1}$, whence the lemma. □

• The dual of $\mathfrak{g}$ is denoted by $\mathfrak{g}^*$ and it identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $\langle .,. \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.

• Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. Let denote by $\mathcal{R}$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and let denote by $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$ is denoted by $W(\mathcal{R})$ and the basis of $\mathcal{R}_+$ is denoted by Π. The neutral element of $W(\mathcal{R})$ is denoted by $1_{\mathfrak{h}}$. For α in $\mathcal{R}$, the corresponding root subspace is denoted by $\mathfrak{g}^\alpha$ and a generator x_α of $\mathfrak{g}^\alpha$ is chosen so that $\langle x_\alpha, x_{-\alpha} \rangle = 1$ for all α in $\mathcal{R}$.

• The normalizers of $\mathfrak{b}$ and $\mathfrak{h}$ in G are denoted by B and $N_G(\mathfrak{h})$ respectively. For x in $\mathfrak{b}$, $\overline{x}$ is the element of $\mathfrak{h}$ such that $x - \overline{x}$ is in the nilpotent radical $\mathfrak{u}$ of $\mathfrak{b}$.

• For X an algebraic B-variety, let denote by $G \times_B X$ the quotient of $G \times X$ under the right action of B given by $(g, x).b := (gb, b^{-1}.x)$. More generally, for k positive integer and for X an algebraic B^k-variety, let denote by $G^k \times_{B^k} X$ the quotient of $G^k \times X$ under the right action of B^k given by $(g,x).b := (gb, b^{-1}.x)$ with g and b in G^k and B^k respectively.

Lemma 1.4. *Let P and Q be parabolic subgroups of G such that P is contained in Q. Let X be a Q-variety and let Y be a closed subset of X, invariant under P. Then $Q.Y$ is a closed subset of X. Moreover, the canonical map from $Q \times_P Y$ to $Q.Y$ is a projective morphism.*

Proof. Since P and Q are parabolic subgroups of G and since P is contained in Q, Q/P is a projective variety. Let denote by $Q \times_P X$ and $Q \times_P Y$ the quotients of $Q \times X$ and $Q \times Y$ under the right action of P given by $(g,x).p := (gp, p^{-1}.x)$. Let $g \mapsto \overline{g}$ be the quotient map from Q to Q/P. Since X is a Q-variety, the map

$$Q \times X \longrightarrow Q/P \times X \qquad (g,x) \longmapsto (\overline{g}, g.x)$$

defines through the quotient an isomorphism from $Q \times_P X$ to $Q/P \times X$. Since Y is a P-invariant closed subset of X, $Q \times_P Y$ is a closed subset of $Q \times_P X$ and its image by the above isomorphism equals $Q/P \times Q.Y$. Hence $Q.Y$ is a closed subset of X since Q/P is a projective variety. From the commutative diagram

$$\begin{tikzcd} Q \times_P Y \arrow[r] \arrow[dr] & Q/P \times Q.Y \arrow[d] \\ & Q.Y \end{tikzcd}$$

one deduces that the map $Q \times_P Y \to Q.Y$ is a projective morphism. $\square$

• For $k \geq 1$ and for the diagonal action of B in $\mathfrak{b}^k$, $\mathfrak{b}^k$ is a B-variety. The canonical map from $G \times \mathfrak{b}^k$ to $G \times_B \mathfrak{b}^k$ is denoted by $(g, x_1, \ldots, x_k) \mapsto \overline{(g, x_1, \ldots, x_k)}$. Let $\mathcal{B}^{(k)}$ and $\mathcal{N}^{(k)}$ be the images of $G \times \mathfrak{b}^k$ and $G \times \mathfrak{u}^k$ respectively by the map $(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$ so that $\mathcal{B}^{(k)}$ and $\mathcal{N}^{(k)}$ are closed subsets of $\mathfrak{g}^k$ by Lemma 1.4. Let $\mathcal{B}_{\rm n}^{(k)}$ be the normalization of $\mathcal{B}^{(k)}$ and let η be the normalization morphism. One has a commutative diagram:

$$\begin{tikzcd} G \times_B \mathfrak{b}^k \arrow[rr, "\gamma_{\rm n}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\rm n}^{(k)} \arrow[dl, "\eta"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

Let $\mathcal{N}_{\rm n}^{(k)}$ be the normalization of $\mathcal{N}^{(k)}$ and let $\varkappa$ be the normalization morphism. One has a commutative diagram:

$$\begin{tikzcd} G \times_B \mathfrak{u}^k \arrow[rr, "\upsilon_{\rm n}"] \arrow[dr, "\upsilon"'] & & \mathcal{N}_{\rm n}^{(k)} \arrow[dl, "\varkappa"] \\ & \mathcal{N}^{(k)} & \end{tikzcd}$$

with υ the restriction of γ to $G \times_B \mathfrak{u}^k$.

• Let i be the injection $(x_1, \ldots, x_k) \mapsto \overline{(1_{\mathfrak{g}}, x_1, \ldots, x_k)}$ from $\mathfrak{b}^k$ to $G \times_B \mathfrak{b}^k$. Then $\iota := \gamma \circ i$ and $\iota_{\rm n} := \gamma_{\rm n} \circ i$ are closed embeddings of $\mathfrak{b}^k$ into $\mathcal{B}^{(k)}$ and $\mathcal{B}_{\rm n}^{(k)}$ respectively. In particular, $\mathcal{B}^{(k)} = G.\iota(\mathfrak{b}^k)$ and $\mathcal{B}_{\rm n}^{(k)} = G.\iota_{\rm n}(\mathfrak{b}^k)$.

• Let e be the sum of the x_β's, β in Π, and let h be the element of $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ such that $\beta(h) = 2$ for all β in Π. Then there exists a unique f in $[\mathfrak{g}, \mathfrak{g}]$ such that (e, h, f) is a principal $\mathfrak{sl}_2$-triple. The one parameter subgroup of G generated by $\mathrm{ad}\,h$ is denoted by $t \mapsto h(t)$. The Borel subalgebra containing f is denoted by $\mathfrak{b}_-$ and its nilpotent radical is denoted by $\mathfrak{u}_-$. Let B_- be the normalizer of $\mathfrak{b}_-$ in G and let U and U_- be the unipotent radicals of B and B_- respectively.

Lemma 1.5. *Let $k \geq 2$ be an integer. Let X be an affine variety and let set $Y := \mathfrak{b}^k \times X$. Let Z be a closed B-invariant subset of Y for the group action given by $g.(v_1, \ldots, v_k, x) = (g(v_1), \ldots, g(v_k), x)$ with $(g, v_1, \ldots, v_k)$ in $B \times \mathfrak{b}^k$ and x in X. Then $Z \cap \mathfrak{h}^k \times X$ is the image of Z by the projection $(v_1, \ldots, v_k, x) \mapsto (\overline{v_1}, \ldots, \overline{v_k}, x)$.*

Proof. For all v in $\mathfrak{b}$,

$$\overline{v} = \lim_{t \to 0} h(t)(v)$$

whence the lemma since Z is closed and B-invariant. □

• For $x \in \mathfrak{g}$, let x_{s} and x_{n} be the semisimple and nilpotent components of x in $\mathfrak{g}$. Let denote by $\mathfrak{g}^x$ and G^x the centralizers of x in $\mathfrak{g}$ and G respectively. For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$ and for A a subgroup of G, let set:

$$\mathfrak{a}^x := \mathfrak{a} \cap \mathfrak{g}^x \qquad A^x := A \cap G^x$$

The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\mathrm{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

and let denote by $\mathfrak{g}_{\mathrm{reg,ss}}$ the set of regular semisimple elements of $\mathfrak{g}$. Both $\mathfrak{g}_{\mathrm{reg}}$ and $\mathfrak{g}_{\mathrm{reg,ss}}$ are G-invariant dense open subsets of $\mathfrak{g}$. Setting $\mathfrak{h}_{\mathrm{reg}} := \mathfrak{h} \cap \mathfrak{g}_{\mathrm{reg}}$, $\mathfrak{b}_{\mathrm{reg}} := \mathfrak{b} \cap \mathfrak{g}_{\mathrm{reg}}$, $\mathfrak{u}_{\mathrm{reg}} := \mathfrak{u} \cap \mathfrak{g}_{\mathrm{reg}}$, $\mathfrak{g}_{\mathrm{reg,ss}} = G(\mathfrak{h}_{\mathrm{reg}})$, $\mathfrak{g}_{\mathrm{reg}} = G(\mathfrak{b}_{\mathrm{reg}})$ and $G(\mathfrak{u}_{\mathrm{reg}})$ is the set of regular elements in the nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ of $\mathfrak{g}$.

Lemma 1.6. *Let $k \geq 2$ be an integer and let x be in $\mathfrak{g}^k$. For O open subset of $\mathfrak{g}_{\mathrm{reg}}$, $P_x \cap O$ is not empty if and only if for some g in $\mathrm{GL}_k(\Bbbk)$, the first component of $g.x$ is in O.*

Proof. Since the components of $g.x$ are in P_x for all g in $\mathrm{GL}_k(\Bbbk)$, the condition is sufficient. Let suppose that $P_x \cap O$ is not empty and let denote by $x_1, \ldots, x_k$ the components of x. For some $(a_1, \ldots, a_k)$ in $\Bbbk^k \setminus \{0\}$,

$$a_1 x_1 + \cdots + a_k x_k \in O$$

Let i be such that $a_i \neq 0$ and let τ be the transposition of $\mathfrak{S}_k$ such that $\tau(1) = i$. Denoting by g the element of $\mathrm{GL}_k(\Bbbk)$ such that $g_{1,j} = a_{\tau(j)}$ for $j = 1, \ldots, k$, $g_{j,j} = 1$ for $j = 2, \ldots, k$ and $g_{j,l} = 0$ for $j \geq 2$ and $j \neq l$, the first component of $g\tau.x$ is in O. □

• Let denote by $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the algebra of $\mathfrak{g}$-invariant elements of $\mathrm{S}(\mathfrak{g})$. Let $p_1, \ldots, p_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degree $d_1, \ldots, d_\ell$ respectively. Let choose the polynomials $p_1, \ldots, p_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \ldots, d_\ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, let consider a shift of p_i in direction y: $p_i(x + ty)$ with $t \in \Bbbk$. Expanding $p_i(x + ty)$ as a polynomial in t, one obtains

$$p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y) t^m; \qquad \forall (t, x, y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{1}$$

where $y \mapsto (m!) p_i^{(m)}(x, y)$ is the derivate at x of p_i at the order m in the direction y. The elements $p_i^{(m)}$ defined by (1) are invariant elements of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g})$ under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. Let remark that $p_i^{(0)}(x, y) = p_i(x)$ while $p_i^{(d_i)}(x, y) = p_i(y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Remark 1.7. The family $\mathcal{P}_x := \{p_i^{(m)}(x,.);\ 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $\mathrm{S}(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. One says that the family $\mathcal{P}_x$ is constructed by the *argument shift method.*

• Let $i \in \{1, \dots, \ell\}$ be. For x in $\mathfrak{g}$, let denote by $\varepsilon_i(x)$ the element of $\mathfrak{g}$ given by

$$\langle \varepsilon_i(x), y\rangle = \frac{\mathrm{d}}{\mathrm{d}t} p_i(x+ty)\,|_{t=0}$$

for all y in $\mathfrak{g}$. Thereby, ε_i is an invariant element of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ under the canonical action of G. According to [Ko63, Theorem 9], for x in $\mathfrak{g}$, x is in $\mathfrak{g}_{\mathrm{reg}}$ if and only if $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ are linearly independent. In this case, $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$.

Let denote by $\varepsilon_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ defined by the equality:

$$\varepsilon_i(x+ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x,y)t^m, \qquad \forall (t,x,y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{2}$$

and let set:

$$V_{x,y} := \mathrm{span}(\{\varepsilon_i^{(0)}(x,y), \dots, \varepsilon_i^{(d_i-1)}(x,y),\ i = 1, \dots, \ell\})$$

for (x,y) in $\mathfrak{g}\times\mathfrak{g}$. According to [Bol91, Corollary 2], $V_{x,y}$ has dimension $\mathrm{b}_{\mathfrak{g}}$ if and only if $P_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$.

1.2. **Main result.** By definition, $\mathcal{B}^{(k)}$ is the subset of elements $(x_1, \dots, x_k)$ of $\mathfrak{g}^k$ such that $x_1, \dots, x_k$ are in a same Borel subalgebra of $\mathfrak{g}$. This subset of $\mathfrak{g}^k$ is closed and contains two interesting subsets: the generalized commuting variety of $\mathfrak{g}$, denoted by $\mathcal{C}^{(k)}$ and the nullcone of $\mathfrak{g}^k$ denoted by $\mathcal{N}^{(k)}$. According to [Mu88, Ch.2, §1, Theorem], for $(x_1, \dots, x_k)$ in $\mathcal{B}^{(k)}$, $(x_1, \dots, x_k)$ is in $\mathcal{N}^{(k)}$ if and only if $x_1, \dots, x_k$ are nilpotent. By definition, $\mathcal{C}^{(k)}$ is the closure in $\mathfrak{g}^k$ of the set of elements whose all components are in a same Cartan subalgebra. According to a Richardson Theorem [Ri79], $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$.

There is a natural projective morphism $G\times_B \mathfrak{b}^k \to \mathcal{B}^{(k)}$. For $k = 1$, this morphism is not birational but for $k \geq 2$, it is birational. Furthermore, denoting by $\mathcal{X}$ the subvariety of elements (x,y) of $\mathfrak{g}\times\mathfrak{h}$ such that y is in the closure of the orbit of x under G, the canonical morphism $G\times_B \mathfrak{b} \to \mathcal{X}$ is projective and birational and $\mathfrak{g}$ is the categorical quotient of $\mathcal{X}$ under the action of $W(\mathcal{R})$ on the factor $\mathfrak{h}$. For $k \geq 2$, the inverse image of $\mathcal{B}^{(k)}$ by the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$ is not irreducible but the canonical action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$ induces a simply transitive action on the set of its irreducible components. Denoting by $\mathcal{B}_{\mathcal{X}}^{(k)}$ one of these components, one has a commutative diagram

$$\begin{tikzcd} G\times_B \mathfrak{b}^k \arrow[rr] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\mathcal{X}}^{(k)} \arrow[dl, "\varpi"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

with ϖ the restriction to $\mathcal{B}_{\mathcal{X}}^{(k)}$ of the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. The first main theorem of this note is the following theorem:

Theorem 1.8. (i) *The variety* $\mathcal{N}^{(k)}$ *has rational singularities.*

(ii) *The variety* $\mathcal{B}_{\mathrm{n}}^{(k)}$ *has rational singularities. Moreover, for* $k \geq 2$, $\mathcal{B}_{\mathfrak{X}}^{(k)}$ *is the normalization of* $\mathcal{B}^{(k)}$ *and* ϖ *is the normalization morphism.*

(iii) *The restriction of* η *to* $\eta^{-1}(\mathcal{N}^{(k)})$ *is an isomorphism onto* $\mathcal{N}^{(k)}$ *and the ideal of definition of* $\eta^{-1}(\mathcal{N}^{(k)})$ *in* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ *is generated by the homogeneous elements of positive degree of* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$.

From Theorem 1.8, one deduces that for $k \geq 2$, the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$ is not generated by the homogeneous elements of positive degree of $\Bbbk[\mathcal{B}^{(k)}]^G$. Moreover, according to a Joseph's result [J07], $\Bbbk[\mathcal{B}^{(k)}]^G$ is isomorphic to $\mathrm{S}(\mathfrak{h}^k)^{W(\mathcal{R})}$ for the diagonal action of $W(\mathcal{R})$ in $\mathfrak{h}^k$.

In the study of the generalized commuting variety, the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of G plays an important role. Denoting by X the closure in $\mathrm{Gr}_\ell(\mathfrak{b})$ of the orbit of $\mathfrak{h}$ under B, $G.X$ is the closure of the orbit of $G.\mathfrak{h}$ and one has the following result:

Theorem 1.9. *Let* X' *be the set of centralizers of regular elements of* $\mathfrak{g}$ *whose semisimple components is regular or subregular. Let* X_{n} *and* $(G.X)_{\mathrm{n}}$ *be the normalizations of* X *and* $G.X$ *respectively. Let denote by* θ_0 *and* θ *the normalization morphisms* $X_{\mathrm{n}} \rightarrow X$ *and* $(G.X)_{\mathrm{n}} \rightarrow G.X$ *respectively.*

(i) *All element of* X *is a commutative algebraic subalgebra of* $\mathfrak{g}$.

(ii) *For* x *in* $\mathfrak{g}$ *and for* v' *a regular linear form on* $\mathfrak{g}^x$, *the stabilizer of* v', *with respect to the coadjoint action of* $\mathfrak{g}^x$, *is in* $G.X$.

(iii) *For* x *in* $\mathfrak{g}$, *the set of elements of* $G.X$ *containing* x *has dimension at most* $\dim \mathfrak{g}^x - \ell$.

(iv) *The set* X' *is an open subset of* X *and* $X \setminus X'$ *has codimension at least* 2 *in* X.

(v) *All irreducible component of* $X \setminus B.\mathfrak{h}$ *has a nonempty intersection with* X'.

(vi) *The set* $G.X'$ *is an open subset of* $G.X$ *and* $G.X \setminus G.X'$ *has codimension at least* 2 *in* $G.X$.

(vii) *All irreducible component of* $G.X \setminus G.\mathfrak{h}$ *has a nonempty intersection with* $G.X'$.

(viii) *The restriction of* θ *to* $\theta^{-1}(G.X')$ *is a homeomorphism onto* $G.X'$ *and* $\theta^{-1}(G.X')$ *is a smooth open subset of* $G.X_{\mathrm{n}}$.

(ix) *The restriction of* θ_0 *to* $\theta_0^{-1}(X')$ *is a homeomorphism onto* X' *and* $\theta_0^{-1}(X')$ *is a smooth open subset of* X_{n}.

Let $\mathfrak{X}_{0,k}$ be the closure in $\mathfrak{b}^k$ of $B.\mathfrak{h}^k$ and let Γ be a desingularization of X in the category of B-varieties. Let $E^{(k)}$ be the inverse image of the canonical vector bundle over X. Then $E^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$. Let set: $\mathcal{C}_{\mathrm{n}}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$. The following theorem is the second main result of this note:

Theorem 1.10. (i) *The variety* $\mathfrak{X}_{0,k}$ *has rational singularities.*

(ii) *The variety* $\mathcal{C}_{\mathrm{n}}^{(k)}$ *is irreducible and* $G \times_B E^{(k)}$ *is a desingularization of* $\mathcal{C}_{\mathrm{n}}^{(k)}$.

(iii) *The normalization morphisms of* $\mathcal{C}_{\mathrm{n}}^{(k)}$ *and* $\mathcal{C}^{(k)}$ *are homeomorphisms.*

(iv) *For* $k = 2$, *the normalizations of* $\mathcal{C}_{\mathrm{n}}^{(k)}$ *and* $\mathcal{C}^{(k)}$ *have rational singularities.*

The proof of Assertion (iv) is an easy consequence of the proof of Assertion (i), and the deep result of Ginzburg [Gi11] which asserts that the normalization of $\mathcal{C}_{\mathrm{n}}^{(2)}$ is Gorenstein.

2. Cohomological results

Let $k \geq 2$ be an integer. According to the above notations, one has the commutative diagrams:

$$\begin{tikzcd} G \times_B \mathfrak{b}^k \arrow[rr, "\gamma_{\mathrm{n}}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\mathrm{n}}^{(k)} \arrow[dl, "\eta"] \\ & \mathcal{B}^{(k)} & \end{tikzcd} \qquad \begin{tikzcd} G \times_B \mathfrak{u}^k \arrow[rr, "\upsilon_{\mathrm{n}}"] \arrow[dr, "\upsilon"'] & & \mathcal{N}_{\mathrm{n}}^{(k)} \arrow[dl, "\varkappa"] \\ & \mathcal{N}^{(k)} & \end{tikzcd}$$

2.1. Since the Borel subalgebras of $\mathfrak{g}$ are conjugate under G, $\mathcal{B}^{(k)}$ is the subset of elements of $\mathfrak{g}^k$ whose components are in a same Borel subalgebra and $\mathcal{N}^{(k)}$ are the elements of $\mathcal{B}^{(k)}$ whose all the components are nilpotent.

Lemma 2.1. (i) *The morphism γ from $G \times_B \mathfrak{b}^k$ to $\mathcal{B}^{(k)}$ is projective and birational. In particular, $G \times_B \mathfrak{b}^k$ is a desingularization of $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(k)}$ has dimension $k\mathrm{b}_{\mathfrak{g}} + n$.*

(ii) *The morphism υ from $G \times_B \mathfrak{u}^k$ to $\mathcal{N}^{(k)}$ is projective and birational. In particular, $G \times_B \mathfrak{u}^k$ is a desingularization of $\mathcal{N}^{(k)}$ and $\mathcal{N}^{(k)}$ has dimension $(k+1)n$.*

Proof. (i) According to Lemma 1.4, γ is a projective morphism. For $1 \leq i < j \leq k$, let $\Omega_{i,j}^{(k)}$ be the inverse image of $\Omega_{\mathfrak{g}}$ by the projection

$$(x_1, \ldots, x_k) \longmapsto (x_i, x_j)$$

Then $\Omega_{i,j}^{(k)}$ is an open subset of $\mathfrak{g}^k$ whose intersection with $\mathcal{B}^{(k)}$ is not empty. Let $\Omega_{\mathfrak{g}}^{(k)}$ be the union of the $\Omega_{i,j}^{(k)}$. According to [Bol91, Corollary 2] and [Ko63, Theorem 9], for (x, y) in $\Omega_{\mathfrak{g}} \cap \mathcal{B}^{(2)}$, $V_{x,y}$ is the unique Borel subalgebra of $\mathfrak{g}$ containing x and y so that the restriction of γ to $\gamma^{-1}(\Omega_{\mathfrak{g}}^{(k)})$ is a bijection onto $\Omega_{\mathfrak{g}}^{(k)}$. Hence γ is birational. Moreover, $G \times_B \mathfrak{b}^k$ is a smooth variety as a vector bundle over the smooth variety G/B, whence the assertion since $G \times_B \mathfrak{b}^k$ has dimension $k\mathrm{b}_{\mathfrak{g}} + n$.

(ii) According to Lemma 1.4, υ is a projective morphism. Let $\mathcal{N}_{\mathrm{reg}}^{(k)}$ be the subset of elements of $\mathcal{N}^{(k)}$ whose at least one component is a regular element of $\mathfrak{g}$. Then $\mathcal{N}_{\mathrm{reg}}^{(k)}$ is an open subset of $\mathcal{N}^{(k)}$. Since a regular nilpotent element is contained in one and only one Borel subalgebra of $\mathfrak{g}$, the restriction of υ to $\upsilon^{-1}(\mathcal{N}_{\mathrm{reg}}^{(k)})$ is a bijection onto $\mathcal{N}_{\mathrm{reg}}^{(k)}$. Hence υ is birational. Moreover, $G \times_B \mathfrak{u}^k$ is a smooth variety as a vector bundle over the smooth variety G/B, whence the assertion since $G \times_B \mathfrak{u}^k$ has dimension $(k+1)n$. □

Let κ be the map

$$U_- \times \mathfrak{u}_{\mathrm{reg}} \longrightarrow \mathfrak{N}_{\mathfrak{g}} \qquad (g, x) \longmapsto g(x)$$

Lemma 2.2. *Let V be the set of elements of $\mathcal{N}^{(k)}$ whose first component is in $U_-(\mathfrak{u}_{\mathrm{reg}})$ and let V_k be the set of elements x of $\mathcal{N}^{(k)}$ such that $P_x \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty.*

(i) *The image of κ is a smooth open subset of $\mathfrak{N}_{\mathfrak{g}}$ and κ is an ismorphism onto $U_-(\mathfrak{u}_{\mathrm{reg}})$.*

(ii) *The subset V of $\mathcal{N}^{(k)}$ is open.*

(iii) *The open subset V of $\mathcal{N}^{(k)}$ is smooth.*

(iv) *The set V_k is a smooth open subset of $\mathcal{N}^{(k)}$.*

Proof. (i) Since $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of $p_1, \ldots, p_\ell$ in $\mathfrak{g}$, $\mathfrak{N}_{\mathfrak{g}} \cap \mathfrak{g}_{\mathrm{reg}}$ is a smooth open susbet of $\mathfrak{N}_{\mathfrak{g}}$ by [Ko63, Theorem 9]. For (g, x) in $U_- \times \mathfrak{u}_{\mathrm{reg}}$ such that $g(x)$ is in $\mathfrak{u}$, $b^{-1}g$ is in G^x for some b in B since $B(x) = \mathfrak{u}_{\mathrm{reg}}$.

Hence $g = 1_{\mathfrak{g}}$ since G^x is contained in B and since $U_- \cap B = \{1_{\mathfrak{g}}\}$. As a result, κ is an injective morphism from the smooth variety $U_- \times \mathfrak{u}_{\rm reg}$ to the smooth variety $\mathfrak{N}_{\mathfrak{g}} \cap \mathfrak{g}_{\rm reg}$. Hence κ is an open immersion by Zariski Main Theorem [Mu88, §9].

(ii) By definition, V is the intersection of $\mathcal{N}^{(k)}$ and $U_-(\mathfrak{u}_{\rm reg}) \times \mathfrak{N}_{\mathfrak{g}}^{k-1}$. So, by (i), it is an open subset of $\mathcal{N}^{(k)}$.

(iii) Let $(x_1, \dots, x_k)$ be in $\mathfrak{u}^k$ and let g be in G such that $(g(x_1), \dots, g(x_k))$ is in V. Then x_1 is in $\mathfrak{u}_{\rm reg}$ and for some (g', b) in $U_- \times B$, $g'b(x_1) = g(x_1)$. Hence $g^{-1}g'b$ is in G^{x_1} and g is in U_-B since G^{x_1} is contained in B. As a result, the map

$$U_- \times \mathfrak{u}_{\rm reg} \times \mathfrak{u}^{k-1} \longrightarrow V \qquad (g, x_1, \dots, x_k) \longmapsto (g(x_1), \dots, g(x_k))$$

is an isomorphism whose inverse is given by

$$V \longrightarrow U_- \times \mathfrak{u}_{\rm reg} \times \mathfrak{u}^{k-1} \qquad (x_1, \dots, x_k) \longmapsto (\kappa^{-1}(x_1)_1, \kappa^{-1}(x_1)_1(x_1), \dots, \kappa^{-1}(x_1)_1(x_k))$$

with κ^{-1} the inverse of κ and $\kappa^{-1}(x_1)_1$ the component of $\kappa^{-1}(x_1)$ on U_-, whence the assertion since $U_- \times \mathfrak{u}_{\rm reg} \times \mathfrak{u}^{k-1}$ is smooth.

(iv) According to Lemma 1.6, $V_k = {\rm GL}_k(\Bbbk).V$, whence the assertion by (iii). □

Corollary 2.3. (i) *The subvariety* $\mathcal{N}^{(k)} \setminus V_k$ *has codimension* $k + 1$.

(ii) *The restriction of* υ *to* $\upsilon^{-1}(V_k)$ *is an isomorphism onto* V_k.

(iii) *The subset* $\upsilon^{-1}(V_k)$ *is a big open subset of* $G \times_B \mathfrak{u}^k$.

Proof. (i) By definition, $\mathcal{N}^{(k)} \setminus V_k$ is the subset of elements x of $\mathcal{N}^{(k)}$ such that P_x is contained in $\mathfrak{g} \setminus \mathfrak{g}_{\rm reg}$. Hence $\mathcal{N}^{(k)} \setminus V_k$ is contained in the image of $G \times_B (\mathfrak{u} \setminus \mathfrak{u}_{\rm reg})^k$ by υ. Let $(x_1, \dots, x_k)$ be in $\mathfrak{u}^k \cap (\mathcal{N}^{(k)} \setminus V_k)$. Then, for all $(a_1, \dots, a_k)$ in $\Bbbk^k$,

$$\langle x_{-\beta}, a_1x_1 + \cdots a_kx_k \rangle = 0$$

for some β in Π. Since Π is finite, P_x is orthogonal to $x_{-\beta}$ for some β in Π. As a result, the subvariety of Borel subalgebras of $\mathfrak{g}$ containing $x_1, \dots, x_k$ has positive dimension. Hence

$$\dim(\mathcal{N}^{(k)} \setminus V_k) < \dim G \times_B (\mathfrak{u} \setminus \mathfrak{u}_{\rm reg})^k = n + k(n-1)$$

Moreover, for β in Π, denoting by $\mathfrak{u}_\beta$ the orthogonal complement of $\mathfrak{g}^{-\beta}$ in $\mathfrak{u}$, $\upsilon(G \times_B (\mathfrak{u}_\beta)^k)$ is contained in $\mathcal{N}^{(k)} \setminus V_k$ and its dimension equal $(k+1)(n-1)$ since the variety of Borel subalgebras containing $\mathfrak{u}_\beta$ has dimension 1, whence the assertion.

(ii) For x in $\mathcal{N}^{(k)}$, P_x is contained in all Borel subalgebra of $\mathfrak{g}$, containing the components of x. Then the restriction of υ to $\upsilon^{-1}(V_k)$ is injective since all regular nilpotent element of $\mathfrak{g}$ is contained in a single Borel subalgebra of $\mathfrak{g}$, whence the assertion by Zariski Main Theorem [Mu88, §9] since V_k is a smooth open subset of $\mathcal{N}^{(k)}$ by Lemma 2.2,(iv).

(iii) Let identify U_- with the open subset U_-B/B of G/B and let denote by ψ the canonical projection from $G \times_B \mathfrak{u}^k$ to G/B. Since $\upsilon^{-1}(V_k)$ is G-invariant, it suffices to prove that $\upsilon^{-1}(V_k) \cap \psi^{-1}(U_-)$ is a big open subset of $\psi^{-1}(U_-)$.

The open subset $\psi^{-1}(U_-)$ of $G \times_B \mathfrak{u}^k$ identifies with $U_- \times \mathfrak{u}^k$ and $\upsilon^{-1}(V_k) \cap \psi^{-1}(U_-)$ identifies with the set of (g, x) such that $P_x \cap \mathfrak{g}_{\rm reg}$ is not empty. Let denote by V_0 the subset of elements x of $\mathfrak{u}^k$ such that $P_x \cap \mathfrak{g}_{\rm reg}$ is not empty. Then $\mathfrak{u}^k \setminus V_0$ is contained in $(\mathfrak{u} \setminus \mathfrak{u}_{\rm reg})^k$ and has codimension at least 2 in $\mathfrak{u}^k$ since $k \geq 2$. As a result, $U_- \times V_0$ is a big open subset of $U_- \times \mathfrak{u}^k$, wence the assertion. □

Theorem 2.4. *Let $k \geq 2$ be an integer and let $\mathcal{N}_{\mathrm{n}}^{(k)}$ be the normalization of $\mathcal{N}^{(k)}$. Then $\mathcal{N}_{\mathrm{n}}^{(k)}$ has rational singularities.*

Proof. Since $G \times_B \mathfrak{u}^k$ is a desingularization of $\mathcal{N}^{(k)}$ by Lemma 2.1,(ii), one has a commutative diagram

$$\begin{tikzcd} G \times_B \mathfrak{u}^k \arrow[rr, "\upsilon_{\mathrm{n}}"] \arrow[dr, "\upsilon"'] & & \mathcal{N}_{\mathrm{n}}^{(k)} \arrow[dl, "\varkappa"] \\ & \mathcal{N}^{(k)} & \end{tikzcd}$$

with $\varkappa$ the normalization morphism. Moreover, υ_{n} is a projective birational morphism. According to Corollary 2.3, $\varkappa^{-1}(V_k)$ is a smooth big open subset of $\mathcal{N}_{\mathrm{n}}^{(k)}$, $\upsilon^{-1}(V_k)$ is a big open subset of $G \times_B \mathfrak{u}^k$ and the restriction of υ_{n} to $\upsilon^{-1}(V_k)$ is an isomorphism onto $\varkappa^{-1}(V_k)$. Hence, by Proposition C.2, with $Y = G \times_B \mathfrak{u}^k$, $\mathcal{N}_{\mathrm{n}}^{(k)}$ has rational singularities. □

2.2. For E a finite dimensional B-module, let denote by $\mathcal{L}_0(E)$ the sheaf of local sections of the vector bundle $G \times_B E$ over G/B. For (k, l) in $\mathbb{N}^2$, let set:

$$E_k := (\mathfrak{b}^*)^{\otimes k} \qquad E_{k,l} := (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} \mathfrak{u}^{\otimes l}$$

so that E_k and $E_{k,l}$ are B-modules. According to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle .,. \rangle$, the dual of $\mathfrak{u}$ identifies with $\mathfrak{u}_-$ so that $\mathfrak{u}_-$ is a B-module.

Proposition 2.5. *Let k, l be nonnegative integers.*
(i) *For all positive integer i,* $\mathrm{H}^i(G/B, \mathcal{L}_0(\mathfrak{u}_-^{\otimes k})) = 0$.
(ii) *For all positive integer i,* $\mathrm{H}^i(G/B, \mathcal{L}_0(E_k)) = 0$.
(iii) *For all positive integer i,* $\mathrm{H}^{i+l}(G/B, \mathcal{L}_0(E_{k,l})) = 0$.

Proof. (i) First of all, since $\mathrm{H}^j(G/B, \mathcal{O}_{G/B}) = 0$ for all positive integer by Borel-Weil-Bott's Theorem [Dem68], one can suppose $k > 0$. According to the identification of $\mathfrak{u}^*$ and $\mathfrak{u}_-$, $\mathrm{S}(\mathfrak{u}_-^k)$ is the algebra of polynomial functions on $\mathfrak{u}^k$. Then, since $G \times_B \mathfrak{u}^k$ is a vector bundle over G/B, for all nonnegative integer i,

$$\mathrm{H}^i(G \times_B \mathfrak{u}^k, \mathcal{O}_{G \times_B \mathfrak{u}^k}) = \mathrm{H}^i(G/B, \mathcal{L}_0(\mathrm{S}(\mathfrak{u}_-^k))) = \bigoplus_{q \in \mathbb{N}} \mathrm{H}^i(G/B, \mathcal{L}_0(\mathrm{S}^q(\mathfrak{u}_-^k)))$$

According to Theorem 2.4, for $i > 0$, the left hand side equals 0 since $G \times_B \mathfrak{u}^k$ is a desingularization of $\mathcal{N}_{\mathrm{n}}^{(k)}$ by Lemma 2.1,(ii). As a result, for $i > 0$,

$$\mathrm{H}^i(G/B, \mathcal{L}_0(\mathrm{S}^k(\mathfrak{u}_-^k))) = 0$$

The decomposition of $\mathfrak{u}_-^k$ as a direct sum of k copies of $\mathfrak{u}_-$ induces a multigradation of $\mathrm{S}(\mathfrak{u}_-)$ such that each subspace of multidegree $(j_1, \ldots, j_k)$ is a B-submodule. Denoting this subspace by $S_{j_1,\ldots,j_k}$, one has

$$\mathrm{S}^k(\mathfrak{u}_-^k)) = \bigoplus_{\substack{(j_1,\ldots,j_k) \in \mathbb{N}^k \\ j_1+\cdots+j_k=k}} S_{j_1,\ldots,j_k} \text{ and } S_{1,\ldots,1} = \mathfrak{u}_-^{\otimes k}$$

Hence for $i > 0$,

$$0 = \mathrm{H}^i(G/B, \mathcal{L}_0(\mathrm{S}^k(\mathfrak{u}_-^k))) = \bigoplus_{\substack{(j_1,\ldots,j_k) \in \mathbb{N}^k \\ j_1+\cdots+j_k=k}} \mathrm{H}^i(G/B, \mathcal{L}_0(S_{j_1,\ldots,j_k}))$$

whence the assertion.

(ii) Let i be a positive integer. Let prove by induction on j that for $k \geq j$,

$$\mathrm{H}^i(G/B, \mathcal{L}_0(E_j \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)})) = 0 \tag{3}$$

By (i), it is true for $j = 0$. Let suppose $j > 0$ and (3) true for $j-1$ and for all $k \geq j-1$. From the exact sequence of B-modules

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{b}^* \longrightarrow \mathfrak{u}_- \longrightarrow 0$$

one deduces the exact sequence of B-modules

$$0 \longrightarrow E_{j-1} \otimes_{\Bbbk} \mathfrak{h} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)} \longrightarrow E_j \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)} \longrightarrow E_{j-1} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j+1)} \longrightarrow 0$$

whence the exact sequence of $\mathcal{O}_{G/B}$-modules

$$0 \longrightarrow \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{h} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)}) \longrightarrow \mathcal{L}_0(E_j \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)}) \longrightarrow \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j+1)}) \longrightarrow 0$$

since $\mathcal{L}_0$ is an exact functor. From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

$$\begin{aligned} \mathrm{H}^i(G/B, \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{h} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)})) \longrightarrow \mathrm{H}^i(G/B, \mathcal{L}_0(E_j \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)})) \\ \longrightarrow \mathrm{H}^i(G/B, \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j+1)})) \end{aligned}$$

By induction hypothesis, the last term equals 0. Since $\mathfrak{h}$ is a trivial B-module,

$$\begin{aligned} \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{h} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)}) = & \quad \mathfrak{h} \otimes_{\Bbbk} \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)}) \\ \mathrm{H}^i(G/B, \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{h} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)})) = & \quad \mathfrak{h} \otimes_{\Bbbk} \mathrm{H}^i(G/B, \mathcal{L}_0(E_{j-1} \otimes_{\Bbbk} \mathfrak{u}_-^{\otimes(k-j)})) \end{aligned}$$

Then, by induction hypothesis again, the first term of the last exact sequence equals 0, whence Equality (3) and whence the assertion since it is true for $k = 0$ by Borel-Weil-Bott's Theorem.

(iii) Let k be a nonnegative integer. Let prove by induction on j that for $i > 0$ and for $l \geq j$,

$$\mathrm{H}^{i+j}(G/B, \mathcal{L}_0(E_{k+l-j,j})) = 0 \tag{4}$$

By (ii) it is true for $j = 0$. Let suppose $j > 0$ and (4) true for $j-1$ and for all $l \geq j-1$. From the short exact sequence of B-modules

$$0 \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{b}^* \longrightarrow 0$$

one deduces the short exact sequence of B-modules

$$0 \longrightarrow E_{k+l-j,j} \longrightarrow \mathfrak{g} \otimes_{\Bbbk} E_{k+l-j,j-1} \longrightarrow E_{k+l-j+1,j-1} \longrightarrow 0$$

whence the exact sequence of $\mathcal{O}_{G/B}$-modules

$$0 \longrightarrow \mathcal{L}_0(E_{k+l-j,j}) \longrightarrow \mathcal{L}_0(\mathfrak{g} \otimes_{\Bbbk} E_{k+l-j,j-1}) \longrightarrow \mathcal{L}_0(E_{k+l-j+1,j-1}) \longrightarrow 0$$

since $\mathcal{L}_0$ is an exact functor. From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

$$\begin{aligned} \mathrm{H}^{i+j-1}(G/B, \mathcal{L}_0(E_{k+l-j+1,j-1})) \longrightarrow \mathrm{H}^{i+j}(G/B, \mathcal{L}_0(E_{k+l-j,j})) \\ \longrightarrow \mathrm{H}^{i+j}(G/B, \mathcal{L}_0(\mathfrak{g} \otimes_{\Bbbk} E_{k+l-j,j-1})) \end{aligned}$$

for all positive integer i. By induction hypothesis, the first term equals 0 for all $i > 0$. Since $\mathfrak{g}$ is a G-module,

$$\begin{aligned} \mathcal{L}_0(\mathfrak{g} \otimes_{\Bbbk} E_{k+l-j,j-1}) &= \mathfrak{g} \otimes_{\Bbbk} \mathcal{L}_0(E_{k+l-j,j-1}) \\ \mathrm{H}^{i+j}(G/B, \mathcal{L}_0(\mathfrak{g} \otimes_{\Bbbk} E_{k+l-j,j-1})) &= \mathfrak{g} \otimes_{\Bbbk} \mathrm{H}^{i+j}(G/B, \mathcal{L}_0(E_{k+l-j,j-1})) \end{aligned}$$

Then by induction hypothesis again, the last term of the last exact sequence equals 0, whence Equality (4) and whence the assertion for $j = l$. □

Corollary 2.6. *Let V be a subspace of $\mathfrak{b}$ containining $\mathfrak{u}$ and let i be a positive integer.*
(i) *For all nonnegative integers k, l,* $\mathrm{H}^{i+l}(G/B, \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes l})) = 0$.
(ii) *For all nonnegative integer m and for all positive integer k,*

$$\mathrm{H}^{i+m}(G/B, \mathcal{L}_0(\textstyle\bigwedge^m(V^k))) = 0$$

Proof. (i) Let prove by induction on j that for $l \geq j$,

$$\mathrm{H}^{i+l}(G/B, \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes j} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)})) = 0 \tag{5}$$

According to Proposition 2.5,(iii), it is true for $j = 0$. Let suppose that it is true for $j-1$. From the exact sequence of B-modules

$$0 \longrightarrow \mathfrak{u} \longrightarrow V \longrightarrow V/\mathfrak{u} \longrightarrow 0$$

one deduces the exact sequence of B-modules

$$\begin{aligned} 0 \longrightarrow (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j+1)} \longrightarrow (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes j} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)} \\ \longrightarrow V/\mathfrak{u} \otimes_{\Bbbk} (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)} \longrightarrow 0 \end{aligned}$$

whence the exact sequence of $\mathcal{O}_{G/B}$-modules

$$\begin{aligned} 0 \longrightarrow \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j+1)}) \longrightarrow \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes j} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)}) \\ \longrightarrow \mathcal{L}_0(V/\mathfrak{u} \otimes_{\Bbbk} (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)}) \longrightarrow 0 \end{aligned}$$

From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

$$\begin{aligned} \mathrm{H}^{i+l}(G/B, \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j+1)})) \longrightarrow \mathrm{H}^{i+l}(G/B, \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes j} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)})) \\ \longrightarrow \mathrm{H}^{i+l}(G/B, \mathcal{L}_0(V/\mathfrak{u} \otimes_{\Bbbk} (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)})) \end{aligned}$$

By induction hypothesis, the first term equals 0. Since $V/\mathfrak{u}$ is a trivial B-module,

$$\mathcal{L}_0(V/\mathfrak{u} \otimes_{\Bbbk} (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)}) = V/\mathfrak{u} \otimes_{\Bbbk} \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)})$$

$$\begin{aligned} \mathrm{H}^{i+l}(G/B, \mathcal{L}_0(V/\mathfrak{u} \otimes_{\Bbbk} (\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)})) = \\ V/\mathfrak{u} \otimes_{\Bbbk} \mathrm{H}^{i+l}(G/B, \mathcal{L}_0((\mathfrak{b}^*)^{\otimes k} \otimes_{\Bbbk} V^{\otimes(j-1)} \otimes_{\Bbbk} \mathfrak{u}^{\otimes(l-j)})) \end{aligned}$$

Then, by induction hypothesis again, the last term of the last exact sequence equals 0, whence Equality (5) and whence the assertion for $j = l$.

(ii) Since

$$\bigwedge^m(V^k) = \bigoplus_{\substack{(j_1,\dots,j_k)\in\mathbb{N}^k \\ j_1+\cdots+j_k=m}} \bigwedge^{j_1}(V)\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}\bigwedge^{j_k}(V)$$

(ii) results from (i) and Proposition B.2. □

3. On the varieties $\mathcal{B}^{(k)}$.

Let $\mathcal{X}$ be the closed subvariety of $\mathfrak{g}\times\mathfrak{h}$ such that $\Bbbk[\mathcal{X}] = \mathrm{S}(\mathfrak{g})\otimes_{\mathrm{S}(\mathfrak{h})^{W(\mathcal{R})}}\mathrm{S}(\mathfrak{h})$. Let $k\geq 2$ be an integer and let $\mathcal{B}^{(k)}_{\mathrm{n}}$ be the normalization of $\mathcal{B}^{(k)}$. The goal of the section is to prove that $\mathcal{B}^{(k)}_{\mathrm{n}}$ is a closed subvariety of $\mathcal{X}^k$ and to give some consequences of this fact.

3.1. According to the notations of Subsection 1.1, γ is the morphism from $G\times_B\mathfrak{b}$ to $\mathfrak{g}$ defined by the map $(g,x)\mapsto g(x)$ through the quotient map.

Lemma 3.1. (i) *The subvariety $\mathcal{X}$ of $\mathfrak{g}\times\mathfrak{h}$ is invariant under the G-action on the first factor and the $W(\mathcal{R})$-action on the second factor. Furthermore, these actions commute.*

(ii) *There exists a well defined G-equivariant morphism γ_{n} from $G\times_B\mathfrak{b}$ to $\mathcal{X}$ such that γ is the compound of γ_{n} and the canonical projection from $\mathcal{X}$ to $\mathfrak{g}$.*

(iii) *The variety $\mathcal{X}$ is irreducible and the morphism γ_{n} is projective and birational.*

(iv) *The variety $\mathcal{X}$ is normal. Moreover, all element of $\mathfrak{g}_{\mathrm{reg}}\times\mathfrak{h}\cap\mathcal{X}$ is a smooth point of $\mathcal{X}$.*

(v) *The algebra $\Bbbk[\mathcal{X}]$ is the space of global sections of $\mathcal{O}_{G\times_B\mathfrak{b}}$ and $\Bbbk[\mathcal{X}]^G = \mathrm{S}(\mathfrak{h})$.*

Proof. (i) By definition, for (x,y) in $\mathfrak{g}\times\mathfrak{h}$, (x,y) is in $\mathcal{X}$ if and only if $p(x)=p(y)$ for all p in $\mathrm{S}(\mathfrak{g})^G$. Hence $\mathcal{X}$ is invariant under the G-action on the first factor and the $W(\mathcal{R})$-action on the second factor. Moreover, these two actions commute.

(ii) Since the map $(g,x)\mapsto(g(x),\overline{x})$ is constant on the B-orbits, there exists a uniquely defined morphism γ_{n} from $G\times_B\mathfrak{b}$ to $\mathfrak{g}\times\mathfrak{h}$ such that $(g(x),\overline{x})$ is the image by γ_{n} of the image of (g,x) in $G\times_B\mathfrak{b}$. The image of γ_{n} is contained in $\mathcal{X}$ since for all p in $\mathrm{S}(\mathfrak{g})^G$, $p(\overline{x}) = p(x) = p(g(x))$. Furthermore, γ_{n} verifies the condition of the assertion.

(iii) According to Lemma 1.4, γ_{n} is a projective morphism. Let (x,y) be in $\mathfrak{g}\times\mathfrak{h}$ such that $p(x)=p(y)$ for all p in $\mathrm{S}(\mathfrak{g})^G$. For some g in G, $g(x)$ is in $\mathfrak{b}$ and its semisimple component is y so that (x,y) is in the image of γ_{n}. As a result, $\mathcal{X}$ is irreducible as the image of the irreducible variety $G\times_B\mathfrak{b}$. Since for all (x,y) in $\mathcal{X}\cap\mathfrak{h}_{\mathrm{reg}}\times\mathfrak{h}_{\mathrm{reg}}$, there exists a unique w in $W(\mathcal{R})$ such that $y=w(x)$, the fiber of γ_{n} at any element $\mathcal{X}\cap G.(\mathfrak{h}_{\mathrm{reg}}\times\mathfrak{h}_{\mathrm{reg}})$ has one element. Hence γ_{n} is birational, whence the assertion.

(iv) Let I be the ideal of $\Bbbk[\mathfrak{g}\times\mathfrak{h}]$ generated by the functions $(x,y)\mapsto p_i(x)-p_i(y)$, $i=1,\dots,\ell$ and let $\mathcal{X}_I$ be the subscheme of $\mathfrak{g}\times\mathfrak{h}$ defined by I. Then $\mathcal{X}$ is the nullvariety of I in $\mathfrak{g}\times\mathfrak{h}$. Since $\mathcal{X}$ has codimension ℓ in $\mathfrak{g}\times\mathfrak{h}$, $\mathcal{X}_I$ is a complete intersection. Let (x,y) be in $\mathcal{X}$ such that x is a regular element of $\mathfrak{g}$ and let $T_{x,y}$ be the tangent space at (x,y) of $\mathcal{X}_I$. For $i=1,\dots,\ell$, the differential at (x,y) of the function $(x,y)\mapsto p_i(x)-p_i(y)$ is

$$(v,w)\mapsto\langle\varepsilon_i(x),v\rangle - \langle\varepsilon_i(y),w\rangle$$

For w in $\mathfrak{h}$, if (v,w) and (v',w) are in $T_{x,y}$ then $v-v'$ is orthogonal to $\varepsilon_1(x),\dots,\varepsilon_\ell(x)$. Since x is regular, $\varepsilon_1(x),\dots,\varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$ by [Ko63, Theorem 9]. Hence

$$\dim T_{x,y}\leq\dim\mathfrak{g}-\ell+\dim\mathfrak{h}$$

As a result, $\mathcal{X} \cap \mathfrak{g}_{\rm reg} \times \mathfrak{h}$ is contained in the subset of smooth points of $\mathcal{X}_I$. According to [V72], $\mathfrak{g} \setminus \mathfrak{g}_{\rm reg}$ has codimension 3 in $\mathfrak{g}$. Hence $\mathcal{X}_I$ is regular in codimension 1 and according to Serre's normality criterion [Bou98, §1, no 10, Théorème 4], $\mathcal{X}_I$ is normal. In particular, I is prime and $\mathcal{X} = \mathcal{X}_I$, whence the assertion.

(v) According to (iii), (iv) and Lemma 1.1, $\Bbbk[\mathcal{X}] = {\rm H}^0(G \times_B \mathfrak{b}, \mathcal{O}_{G\times_B \mathfrak{b}})$. Under the action of G in $\mathfrak{g} \times \mathfrak{h}$, $\Bbbk[\mathfrak{g} \times \mathfrak{h}]^G = {\rm S}(\mathfrak{g})^G \otimes_{\Bbbk} {\rm S}(\mathfrak{h})$ and its image in $\Bbbk[\mathcal{X}]$ by the quotient morphism equals ${\rm S}(\mathfrak{h})$. Moreover, since G is reductive, $\Bbbk[\mathcal{X}]^G$ is the image of $\Bbbk[\mathfrak{g} \times \mathfrak{h}]^G$ by the quotient morphism, whence the assertion. □

The following proposition is given by [He76, Theorem B and Corollary].

Proposition 3.2. (i) *For $i > 0$, ${\rm H}^i(G/B, \mathcal{L}_0({\rm S}(\mathfrak{b}^*))$ equals 0.*

(ii) *The variety $\mathcal{X}$ has rational singularities.*

Corollary 3.3. (i) *Let x and x' be in $\mathfrak{b}_{\rm reg}$ such that $(x', \overline{x'})$ is in $G.(x, \overline{x})$. Then x' is in $B(x)$.*

(ii) *For all w in $W(\mathcal{R})$, the map*

$$U_- \times \mathfrak{b}_{\rm reg} \longrightarrow \mathcal{X} \qquad (g, x) \longmapsto (g(x), w(\overline{x}))$$

is an isomorphism onto a smooth open subset of $\mathcal{X}$.

Proof. (i) The semisimple components of x and x' are conjugate under B since they are conjugate to $\overline{x}$ under B. Let b and b' be in B such that $\overline{x}$ is the semisimple component of $b(x)$ and $b'(x')$. Then the nilpotent components of $b(x)$ and $b'(x')$ are regular nilpotent elements of $\mathfrak{g}^{\overline{x}}$, belonging to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}^{\overline{x}}$ of $\mathfrak{g}^{\overline{x}}$. Hence x' is in $B(x)$.

(ii) Since the action of G and $W(\mathcal{R})$ on $\mathcal{X}$ commute, it suffices to prove the corollary for $w = 1_{\mathfrak{h}}$. Let denote by θ the map

$$U_- \times \mathfrak{b}_{\rm reg} \longrightarrow \mathcal{X} \qquad (g, x) \longmapsto (g(x), \overline{x})$$

Let (g, x) and (g', x') be in $U_- \times \mathfrak{b}_{\rm reg}$ such that $\theta(g, x) = \theta(g', x')$. By (i), $x' = b(x)$ for some b in B. Hence $g^{-1}g'b$ is in G^x. Since x is in $\mathfrak{b}_{\rm reg}$, G^x is contained in B and $g^{-1}g'$ is in $U_- \cap B$, whence $(g, x) = (g', x')$ since $U_- \cap B = \{1_{\mathfrak{g}}\}$. As a result, θ is a dominant injective map from $U_- \times \mathfrak{b}_{\rm reg}$ to the normal variety $\mathcal{X}$. Hence θ is an isomorphism onto a smooth open subset of $\mathcal{X}$, by Zariski Main Theorem [Mu88, §9]. □

3.2. Let Δ be the diagonal of $(G/B)^k$ and let $\mathcal{J}_\Delta$ be its ideal of definition in $\mathcal{O}_{(G/B)^k}$. The variety G/B identifies with Δ so that $\mathcal{O}_{(G/B)^k}/\mathcal{J}_\Delta$ is isomorphic to $\mathcal{O}_{G/B}$. For E a B^k-module, let denote by $\mathcal{L}(E)$ the sheaf of local sections of the vector bundle $G^k \times_{B^k} E$ over $(G/B)^k$.

Lemma 3.4. *Let E be a finite dimensional B^k-module and let $E_\bullet$ be an acyclic complex of finite dimensional B^k-modules. Let denote by $\overline{E}$ the B-module defined by the diagonal action of B on E.*

(i) *The short sequence of $\mathcal{O}_{(G/B)^k}$-modules*

$$0 \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{G^k \times_{B^k} \mathfrak{b}^k}} \mathcal{L}(E) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}_0(\overline{E}) \longrightarrow 0$$

is exact.

(ii) *The complex $\mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E_\bullet)$ is acyclic.*

Proof. (i) Since $\mathcal{L}(E)$ is a locally free $\mathcal{O}_{(G/B)^k}$-module, the short sequence of $\mathcal{O}_{(G/B)^k}$-modules

$$0 \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{O}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \longrightarrow 0$$

is exact, whence the assertion since $\mathcal{O}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E)$ is isomorphic to $\mathcal{L}_0(E)$.

(ii) Since Δ is a smooth subvariety of the smooth variety $(G/B)^k$, it is a locally complete intersection. Hence locally, $\mathcal{J}_\Delta$ has a free resolution by a Koszul complex

$$K_\bullet \longrightarrow \mathcal{J}_\Delta \longrightarrow 0$$

Locally, one has a double complex $C_{\bullet,\bullet} := K_\bullet \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E_\bullet)$. Since $\mathcal{L}(E_\bullet)$ is an acyclic complex of locally free modules, the complex $C_{\bullet,i}$ is acyclic for all i and the complex $C_{i,\bullet}$ is acyclic for all $i > 0$, whence the assertion since the exactness of the complex of the assertion is a local property. □

Corollary 3.5. *Let V be a subspace of $\mathfrak{b}$ containining $\mathfrak{u}$ and let i be a positive integer.*

(i) *For all nonnegative integer m,*

$$\mathrm{H}^{i+m+1}((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\textstyle\bigwedge^m(V^k))) = 0$$

(ii) *For all nonnegative integer m,*

$$\mathrm{H}^{i+1}((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{G/B\times G/B}} \mathcal{L}(\mathrm{S}^m((\mathfrak{b}^*)^k))) = 0$$

Proof. The spaces $(\mathfrak{b}^*)^k$ and V^k are naturally B^k-modules. Then it is so for $\mathrm{S}^m((\mathfrak{b}^*)^k)$ and $\bigwedge^m(V^k)$. Let denote by E one of these two modules and let denote by $\overline{E}$ the B-module defined by the diagonal action of B on E. According to Lemma 3.4,(i), the short sequence of $\mathcal{O}_{(G/B)^k}$-modules

$$0 \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}_0(\overline{E}) \longrightarrow 0$$

is exact whence the cohomology long exact sequence

$$\cdots \longrightarrow \mathrm{H}^j((G/B)^k, \mathcal{L}(E)) \longrightarrow \mathrm{H}^j(G/B, \mathcal{L}_0(\overline{E}))$$
$$\longrightarrow \mathrm{H}^{j+1}((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E)) \longrightarrow \mathrm{H}^{j+1}((G/B)^k, \mathcal{L}(E)) \longrightarrow \cdots$$

Since

$$\textstyle\bigwedge^m(V^k) = \bigoplus_{\substack{(j_1,\ldots,j_k)\in\mathbb{N}^k \\ j_1+\cdots+j_k=m}} \bigwedge^{j_1}(V) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \bigwedge^{j_k}(V)$$
$$\mathrm{S}^m((\mathfrak{b}^*)^k) = \bigoplus_{\substack{(j_1,\ldots,j_k)\in\mathbb{N}^k \\ j_1+\cdots+j_k=m}} \mathrm{S}^{j_1}(\mathfrak{b}^*) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathrm{S}^{j_k}(\mathfrak{b}^*)$$

$\mathrm{H}^j((G/B)^k, \mathcal{L}(E)) = 0$ for $j > m$ and for $E = \bigwedge^m(V^k)$ by Corollary 2.6,(ii) and for $j > 0$ and for $E = \mathrm{S}^m((\mathfrak{b}^*)^k)$ by Proposition 2.5,(ii) and Proposition B.2. As a result, the sequence

$$0 \longrightarrow \mathrm{H}^j(G/B, \mathcal{L}_0(\overline{E})) \longrightarrow \mathrm{H}^{j+1}((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E)) \longrightarrow 0$$

is exact with $j = i + m$ for $E = \bigwedge^m(V^k)$ and with $j = i$ for $E = \mathrm{S}^m((\mathfrak{b}^*)^k)$, whence Assertion (i) by Corollary 2.6,(ii) and Assertion (ii) by Proposition 2.5,(ii) and Proposition B.2. □

Let set $\mathfrak{Y} := G^k \times_{B^k} \mathfrak{b}^k$. The map

$$G \times \mathfrak{b}^k \longrightarrow G^k \times \mathfrak{b}^k \qquad (g, v_1, \ldots, v_k) \longmapsto (g, \ldots, g, v_1, \ldots, v_k)$$

defines through the quotient a closed immersion from $G \times_B \mathfrak{b}^k$ to $\mathfrak{Y}$. Let denote it by ν.

Corollary 3.6. *Let $\mathcal{J}$ be the ideal of definition in $\mathcal{O}_{\mathfrak{Y}}$ of $\nu(G \times_B \mathfrak{b}^k)$. Then $\mathrm{H}^i(\mathfrak{Y}, \mathcal{J}) = 0$ for all positive integer i.*

Proof. Let denote by κ the canonical projection from $\mathfrak{Y}$ to $(G/B)^k$. Then

$$\kappa_*(\mathcal{J}) = \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\mathrm{S}((\mathfrak{b}^*)^k))$$

so that

$$\mathrm{H}^i(\mathfrak{Y}, \mathcal{J}) = \mathrm{H}^i((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\mathrm{S}((\mathfrak{b}^*)^k)))$$

for all i. According to Corollary 3.5,(ii), the both sides equal 0 for $i \geq 2$.

Since $\langle .,. \rangle$ identifies $\mathfrak{g}^*$ and its dual, one has a short exact sequence of B^k-modules:

$$0 \longrightarrow \mathfrak{u}^k \longrightarrow (\mathfrak{g}^*)^k \longrightarrow (\mathfrak{b}^*)^k \longrightarrow 0$$

From this exact sequence, on deduces the exact Koszul complex

$$\cdots \xrightarrow{\mathrm{d}} K_2 \xrightarrow{\mathrm{d}} K_1 \xrightarrow{\mathrm{d}} K_0 \longrightarrow \mathrm{S}((\mathfrak{b}^*)^k) \longrightarrow 0$$

with

$$K_m := \mathrm{S}((\mathfrak{g}^*)^k) \otimes_{\Bbbk} \textstyle\bigwedge^m((\mathfrak{u})^k)$$

$$\mathrm{d} a \otimes a_0 \wedge \cdots \wedge a_m := \sum_{i=0}^{m} (-1)^i a_i a \otimes a_0 \wedge \cdots a_{\hat{i}} \wedge \cdots \wedge a_m$$

This complex $K_\bullet$ is canonically graded by

$$K_\bullet := \sum_q K_\bullet^q \text{ with } K_m^q := \mathrm{S}^{q-m}((\mathfrak{g}^*)^k) \otimes_{\Bbbk} \textstyle\bigwedge^m(\mathfrak{u}^k)$$

so that the sequence

$$\cdots \longrightarrow K_2^q \longrightarrow K_1^q \longrightarrow K_0^q \longrightarrow \mathrm{S}^q((\mathfrak{b}^*)^k) \longrightarrow 0$$

is exact. According to Lemma 3.4,(ii), the sequence of $\mathcal{O}_{(G/B)^k}$-modules:

$$\cdots \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K_2^q) \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K_1^q) \longrightarrow$$
$$\longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K_0^q) \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\mathrm{S}^q((\mathfrak{b}^*)^k)) \longrightarrow 0$$

is exact. Since $\mathrm{H}^\bullet$ is an exact δ-functor, for i nonnegative integer,

$$\mathrm{H}^i((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\mathrm{S}((\mathfrak{b}^*)^k))) = 0$$

if

$$\mathrm{H}^{i+m}((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K_m^q)) = 0$$

for all nonnegative integers q and m. Since $(\mathfrak{g}^*)^k$ is a G^k-module, for all nonnegative integers q and m, $\mathcal{L}(K_m^q)$ is isomorphic to

$$\mathrm{S}^{q-m}((\mathfrak{g}^*)^k) \otimes_{\Bbbk} \mathcal{L}(\textstyle\bigwedge^m((\mathfrak{u})^k))$$

As a result,

$$\mathrm{H}^i((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\mathrm{S}((\mathfrak{b}^*)^k))) = 0$$

if

$$\mathrm{H}^{i+m}((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\textstyle\bigwedge^m((\mathfrak{u})^k))) = 0$$

for all nonnegative integer m. According to Corollary 3.5,(i),

$$\mathrm{H}^{1+m}((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\textstyle\bigwedge^m(\mathfrak{u}^k)) = 0$$

for all positive integer m. From the cohomology long exact sequence deduced from the short exact sequence of Lemma 3.4,(i), with E the trivial module of dimension 1, the sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0((G/B)^k, \mathcal{J}_\Delta) \longrightarrow \mathrm{H}^0((G/B)^k, \mathcal{O}_{(G/B)^k}) \longrightarrow \mathrm{H}^0(G/B, \mathcal{O}_{G/B}) \\ \longrightarrow \mathrm{H}^1((G/B)^k, \mathcal{J}_\Delta) \longrightarrow \mathrm{H}^1((G/B)^k, \mathcal{O}_{(G/B)^k}) \end{aligned}$$

is exact. According to Borel-Weil-Bott's Theorem [Dem68],

$$\mathrm{H}^0((G/B)^k, \mathcal{O}_{(G/B)^k}) = \Bbbk \qquad \mathrm{H}^0(G/B, \mathcal{O}_{G/B}) = \Bbbk \qquad \mathrm{H}^1((G/B)^k, \mathcal{O}_{(G/B)^k}) = 0$$

whence

$$\mathrm{H}^1((G/B)^k, \mathcal{J}_\Delta) = 0$$

As a result,

$$\mathrm{H}^1((G/B)^k, \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\mathrm{S}((\mathfrak{b}^*)^k))) = 0$$

whence the corollary. □

3.3. According to Lemma 2.1,(i), $G\times_B\mathfrak{b}^k$ is a desingularization of $\mathcal{B}^{(k)}$ and one has a commutative diagram

$$\begin{tikzcd} G \times_B \mathfrak{b}^k \arrow[rr, "\gamma_{\mathrm{n}}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\mathrm{n}}^{(k)} \arrow[dl, "\eta"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

Lemma 3.7. *Let ϖ be the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. Let denote by ι_1 the map*

$$\mathfrak{b}^k \longrightarrow \mathcal{X}^k \qquad (x_1, \dots, x_k) \longmapsto (x_1, \dots, x_k, \overline{x_1}, \dots, \overline{x_k})$$

(i) *The map ι_1 is a closed embedding of $\mathfrak{b}^k$ into $\mathcal{X}^k$.*

(ii) *The subvariety $\iota_1(\mathfrak{b}^k)$ of $\mathcal{X}^k$ is an irreducible component of $\varpi^{-1}(\mathfrak{b}^k)$.*

(iii) *The subvariety $\varpi^{-1}(\mathfrak{b}^k)$ of $\mathcal{X}^k$ is invariant under the canonical action of $W(\mathcal{R})^k$ in $\mathcal{X}^k$ and this action induces a simply transitive action of $W(\mathcal{R})^k$ on the set of irreducible components of $\varpi^{-1}(\mathfrak{b}^k)$.*

Proof. (i) The map

$$\mathfrak{b}^k \longrightarrow G^k \times \mathfrak{b}^k \qquad (x_1, \dots, x_k) \longmapsto (1_{\mathfrak{g}}, \dots, 1_{\mathfrak{g}}, x_1, \dots, x_k)$$

defines through the quotient a closed embedding of $\mathfrak{b}^k$ in $G^k \times_{B^k} \mathfrak{b}^k$. Let denote it by ι'. Let $\gamma_{\mathrm{n}}^{(k)}$ be the map

$$G^k \times_{B^k} \mathfrak{b}^k \longrightarrow \mathcal{X}^k \qquad (x_1, \dots, x_k) \longmapsto (\gamma_{\mathrm{n}}(x_1), \dots, \gamma_{\mathrm{n}}(x_k))$$

Then $\iota_1 = \gamma_{\mathrm{n}}^{(k)} \circ \iota'$. Since γ_{n} is a projective morphism, ι_1 is a closed morphism. Moreover, it is injective since $\varpi\circ\iota_1$ is the identity of $\mathfrak{b}^k$.

(ii) According to Lemma 3.1,(ii) and Lemma 1.4, ϖ is a finite morphism. So $\varpi^{-1}(\mathfrak{b}^k)$ and $\mathfrak{b}^k$ have the same dimension. According to (i), $\iota_1(\mathfrak{b}^k)$ is an irreducible subvariety of $\omega^{-1}(\mathfrak{b}^k)$ of the same dimension, whence the assertion.

(iii) Since all the fibers of ϖ are invariant under the action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$, $\varpi^{-1}(\mathfrak{b}^k)$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\varpi^{-1}(\mathfrak{b}^k)$. For w in $W(\mathcal{R})^k$, let set $Z_w := w.\iota_1(\mathfrak{b}^k)$. Then Z_w is an irreducible component of $\varpi^{-1}(\mathfrak{b}^k)$ for all w in $W(\mathcal{R})^k$ by (ii). For w in $W(\mathcal{R})^k$ such that $Z_w = \iota_1(\mathfrak{b}^k)$, for all $(x_1,\dots,x_k)$ in $\mathfrak{h}^k_{\rm reg}$, $(x_1,\dots,x_k,w.(x_1,\dots,x_k))$ is in $\iota_1(\mathfrak{b}^k)$ so that $(x_1,\dots,x_k)$ is invariant under w and w is the identity.

Let Z be an irreducible component of $\varpi^{-1}(\mathfrak{b}^k)$ and let Z_0 be its image by the map

$$(x_1,\dots,x_k,y_1,\dots,y_k) \longmapsto (\overline{x_1},\dots,\overline{x_k},y_1,\dots,y_k)$$

Since ϖ is G^k-equivariant and since $\mathfrak{b}^k$ is invariant under B^k, $\varpi^{-1}(\mathfrak{b}^k)$ and Z are invariant under B^k. Hence by Lemma 1.5, Z_0 is closed. Moreover, since the image of the map

$$Z_0\times\mathfrak{u}^k \longrightarrow \mathcal{X}^k \qquad ((x_1,\dots,x_k,y_1,\dots,y_k),(u_1,\dots,u_k)) \longmapsto (x_1+u_1,\dots,x_k+u_k,y_1,\dots,y_k)$$

is an irreducible subset of $\varpi^{-1}(\mathfrak{b}^k)$ containing Z, Z is the image of this map. Since Z_0 is contained in $\mathcal{X}^k$, Z_0 is contained in the image of the map

$$\mathfrak{h}^k\times W(\mathcal{R})^k \longrightarrow \mathfrak{h}^k\times\mathfrak{h}^k \qquad (x_1,\dots,x_k,w_1,\dots,w_k) \longmapsto (x_1,\dots,x_k,w_1(x_1),\dots,w_k(x_k))$$

Then, since $W(\mathcal{R})$ is finite and since Z_0 is irreducible, for some w in $W(\mathcal{R})^k$, Z_0 is the image of $\mathfrak{h}^k$ by the map

$$(x_1,\dots,x_k) \longmapsto (x_1,\dots,x_k,w.(x_1,\dots,x_k))$$

Then $Z = Z_w$, whence the assertion. □

Let consider the diagonal action of G on $\mathcal{X}^k$ and let identify $G\times_B\mathfrak{b}^k$ with $\nu(G\times_B\mathfrak{b}^k)$ by the closed immersion ν.

Corollary 3.8. *Let set* $\mathcal{B}^{(k)}_{\mathcal{X}} := G.\iota_1(\mathfrak{b}^k)$.

(i) *The subset* $\mathcal{B}^{(k)}_{\mathcal{X}}$ *is the image of* $G\times_B\mathfrak{b}^k$ *by* $\gamma^{(k)}_{\rm n}$. *Moreover, the restriction of* $\gamma^{(k)}_{\rm n}$ *to* $G\times_B\mathfrak{b}^k$ *is a projective birational morphism from* $G\times_B\mathfrak{b}^k$ *onto* $\mathcal{B}^{(k)}_{\mathcal{X}}$.

(ii) *The subset* $\mathcal{B}^{(k)}_{\mathcal{X}}$ *of* $\mathcal{X}^k$ *is an irreducible component of* $\varpi^{-1}(\mathcal{B}^{(k)})$.

(iii) *The subvariety* $\varpi^{-1}(\mathcal{B}^{(k)})$ *of* $\mathcal{X}^k$ *is invariant under* $W(\mathcal{R})^k$ *and this action induces a simply transitive action of* $W(\mathcal{R})^k$ *on the set of irreducible components of* $\varpi^{-1}(\mathcal{B}^{(k)})$.

(iv) *The subalgebra* $\Bbbk[\mathcal{B}^{(k)}]$ *of* $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)}]$ *equals* $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$ *with respect to the action of* $W(\mathcal{R})^k$ *on* $\varpi^{-1}(\mathcal{B}^{(k)})$.

Proof. (i) Let $\gamma_{\mathcal{X}}$ be the restriction of $\gamma^{(k)}_{\rm n}$ to $G\times_B\mathfrak{b}^k$. Since $\iota_1 = \gamma^{(k)}_{\rm n}\circ\iota'$, since $G\times_B\mathfrak{b}^k = G.\iota'(\mathfrak{b}^{(k)})$ and since $\gamma^{(k)}_{\rm n}$ is G-equivariant, $\mathcal{B}^{(k)}_{\mathcal{X}} = \gamma_{\mathcal{X}}(G\times_B\mathfrak{b}^k)$. Hence $\mathcal{B}^{(k)}_{\mathcal{X}}$ is closed in $\mathcal{X}^k$ and $\gamma_{\mathcal{X}}$ is a projective morphism from $G\times_B\mathfrak{b}^k$ to $\mathcal{B}^{(k)}_{\mathcal{X}}$ since $\gamma^{(k)}_{\rm n}$ is a projective morphism. According to Lemma 2.1,(i), $\varpi\circ\gamma_{\mathcal{X}}$ is a birational morphism onto $\mathcal{B}^{(k)}$. Then $\gamma_{\mathcal{X}}$ is birational since $\varpi(\mathcal{B}^{(k)}_{\mathcal{X}}) = \mathcal{B}^{(k)}$, whence the assertion.

(ii) Since ϖ is a finite morphism, $\varpi^{-1}(\mathcal{B}^{(k)})$, $\mathcal{B}^{(k)}_{\mathcal{X}}$ and $\mathcal{B}^{(k)}$ have the same dimension, whence the assertion since $\mathcal{B}^{(k)}_{\mathcal{X}}$ is irreducible as an image of an irreducible variety.

(iii) Since the fibers of ϖ are invariant under $W(\mathcal{R})^k$, $\varpi^{-1}(\mathcal{B}^{(k)})$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\varpi^{-1}(\mathcal{B}^{(k)})$. Let Z be an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$. Since ϖ is G^k-equivariant, $\varpi^{-1}(\mathcal{B}^{(k)})$ and Z are invariant under the diagonal action of G. Moreover,

$Z = G.(Z \cap \varpi^{-1}(\mathfrak{b}^k))$ since $\mathcal{B}^{(k)} = G.\mathfrak{b}^k$. Hence for some irreducible component Z_0 of $Z \cap \varpi^{-1}(\mathfrak{b}^k)$, $Z = G.Z_0$. According to Lemma 3.7,(iii), Z_0 is contained in $w.\iota_1(\mathfrak{b}^k)$ for some w in $W(\mathcal{R})^k$. Hence Z is contained $w.\mathcal{B}_{\mathcal{X}}^{(k)}$ since the actions of G^k and $W(\mathcal{R})^k$ on $\mathcal{X}^k$ commute, whence $Z = w.\mathcal{B}_{\mathcal{X}}^{(k)}$ since Z is an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$.

Let $w = (w_1, \dots, w_k)$ be in $W(\mathcal{R})^k$ such that $w.\mathcal{B}_{\mathcal{X}}^{(k)} = \mathcal{B}_{\mathcal{X}}^{(k)}$. Let x be in $\mathfrak{h}_{\rm reg}$ and let i be equal to $1, \dots, k$. Let set

$$z := (x_1, \dots, x_k, \overline{x_1}, \dots, \overline{x_k}) \text{ with } x_j := \begin{cases} x & \text{if} \quad j = i \\ x_j = e & \text{otherwise} \end{cases}$$

Then there exists $(y_1, \dots, y_k)$ in $\mathfrak{b}^k$ and g in G such that

$$w.z = (g(y_1), \dots, g(y_k), \overline{y_1}, \dots, \overline{y_k})$$

Then, for some b in B, $b(y_i) = \overline{y_i}$ since y_i is a regular semisimple element, belonging to $\mathfrak{b}$. As a result, $gb^{-1}(\overline{y_i}) = x$ and $w_i(x) = \overline{y_i}$. Hence gb^{-1} is an element of $N_G(\mathfrak{h})$ representing w_i^{-1}. Furthermore, since $gb^{-1}(b(y_j)) = e$ for $j \neq i$, $b(y_j)$ is a regular nilpotent element belonging to $\mathfrak{b}$. Then, since there is one and only one Borel subalgebra containing a regular nilpotent element, $gb^{-1}(\mathfrak{b}) = \mathfrak{b}$. Hence $w_i = 1_{\mathfrak{h}}$, whence the assertion.

(iv) Since the fibers of ϖ are invariant under $W(\mathcal{R})^k$, $\Bbbk[\mathcal{B}^{(k)}]$ is contained in $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$. Let p be in $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$. Since $W(\mathcal{R})$ is a finite group, p is the restriction to $\varpi^{-1}(\mathcal{B}^{(k)})$ of an element q of $\Bbbk[\mathcal{X}]^{\otimes k}$, invariant under $W(\mathcal{R})^k$. Since $\Bbbk[\mathcal{X}]^{W(\mathcal{R})} = \mathrm{S}(\mathfrak{g})$, q is in $\mathrm{S}(\mathfrak{g})^{\otimes k}$ and p is in $\Bbbk[\mathcal{B}^{(k)}]$, whence the assertion. □

Let recall that θ is the map

$$U_- \times \mathfrak{b}_{\rm reg} \longrightarrow \mathcal{X} \qquad (g, x) \longmapsto (g(x), \overline{x})$$

and let denote by W'_k the inverse image of $\theta(U_- \times \mathfrak{b}_{\rm reg})$ by the projection

$$\mathcal{B}_{\mathcal{X}}^{(k)} \longrightarrow \mathcal{X} \qquad (x_1, \dots, x_k, y_1, \dots, y_k) \longmapsto (x_1, y_1)$$

Lemma 3.9. *Let W_k be the subset of elements (x, y) of $\mathcal{B}_{\mathcal{X}}^{(k)}$ ($x \in \mathfrak{g}^k, y \in \mathfrak{h}^k$) such that $P_x \cap \mathfrak{g}_{\rm reg}$ is not empty.*

(i) *The subset W'_k of $\mathcal{B}_{\mathcal{X}}^{(k)}$ is a smooth open subset. Moreover, the map*

$$U_- \times \mathfrak{b}_{\rm reg} \times \mathfrak{b}^{k-1} \longrightarrow W'_k \qquad (g, x_1, \dots, x_k) \longmapsto (g(x_1), \dots, g(x_k), \overline{x_1}, \dots, \overline{x_k})$$

is an isomorphism of varieties.

(ii) *The subset $\mathcal{B}_{\mathcal{X}}^{(k)}$ of $\mathfrak{g}^k \times \mathfrak{h}^k$ is invariant under the canonical action of* $\mathrm{GL}_k(\Bbbk)$.

(iii) *The subset W_k of $\mathcal{B}_{\mathcal{X}}^{(k)}$ is a smooth open subset. Moreover, W_k is the $G \times \mathrm{GL}_k(\Bbbk)$-invariant set generated by W'_k.*

(iv) *The subvariety $\mathcal{B}_{\mathcal{X}}^{(k)} \setminus W_k$ has codimension at least $2k$.*

Proof. (i) According to Corollary 3.3,(ii), the image of θ is an open subset of $\mathcal{X}$. Hence W'_k is an open subset of $\mathcal{B}_{\mathcal{X}}^{(k)}$. Let $(x_1, \dots, x_k)$ be in $\mathfrak{b}^k$ and let g be in G such that $(g(x_1), \dots, g(x_k), \overline{x_1}, \dots, \overline{x_k})$ is in W'_k. Then x_1 is in $\mathfrak{b}_{\rm reg}$ and for some g' in U_- and for some x'_1 in $\mathfrak{b}_{\rm reg}$, $g.(x_1, \overline{x_1}) = g'.(x'_1, \overline{x'_1})$. Hence, according to Corollary 3.3,(i), for some b in B, $x_1' = b(x_1)$. So, $g^{-1}g'b$ is in G^{x_1} and g is in U_-B since G^{x_1} is contained in B. As a result, the map

$$U_- \times \mathfrak{b}_{\rm reg} \times \mathfrak{b}^{k-1} \longrightarrow W_k' \qquad (g, x_1, \ldots, x_k) \longmapsto (g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k})$$

is an isomorphism whose inverse is given by

$$\begin{array}{ccc} W_k' & \longrightarrow & U_- \times \mathfrak{b}_{\rm reg} \times \mathfrak{b}^{k-1} \\ (x_1, \ldots, x_k) & \longmapsto & (\theta^{-1}(x_1, \overline{x_1})_1, \theta^{-1}(x_1, \overline{x_1})_1(x_1), \ldots, \theta^{-1}(x_1, \overline{x_1})_1(x_k)) \end{array}$$

with θ^{-1} the inverse of θ and $\theta^{-1}(x_1, \overline{x_1})_1$ the component of $\theta^{-1}(x_1, \overline{x_1})$ on U_-, whence the assertion since $U_- \times \mathfrak{b}_{\rm reg} \times \mathfrak{b}^{k-1}$ is smooth.

(ii) For $(x_1, \ldots, x_k)$ in $\mathfrak{b}^k$ and for $(a_{i,j}, 1 \leq i, j \leq k)$ in $\mathrm{GL}_k(\Bbbk)$,

$$\overline{\sum_{j=1}^{k} a_{i,j} x_j} = \sum_{j=1}^{k} a_{i,j} \overline{x_j}$$

So, $\iota_1(\mathfrak{b}^k)$ is invariant under the action of $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{g}^k \times \mathfrak{h}^k$ defined by

$$(a_{i,j}, 1 \leq i, j \leq k).(x_1, \ldots, x_k, y_1, \ldots, y_k) := (\sum_{j=1}^{k} a_{i,j} x_j, j = 1, \ldots, k, \sum_{j=1}^{k} a_{i,j} y_j, j = 1, \ldots, k)$$

whence the assertion since $\mathcal{B}_{\mathcal{X}}^{(k)} = G.\iota_1(\mathfrak{b}^k)$ and since the actions of G and $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{g}^k \times \mathfrak{h}^k$ commute.

(iii) According to (i), $G.W_k'$ is a smooth open subset of $\mathcal{B}_{\mathcal{X}}^{(k)}$. Moreover, $G.W_k'$ is the subset of elements (x, y) such that the first component of x is regular. So, by (ii) and Lemma 1.6, $W_k = \mathrm{GL}_k(\Bbbk).(G.W_k')$, whence the assertion.

(iv) According to Corollary 3.8,(i), $\mathcal{B}_{\mathcal{X}}^{(k)}$ is the image of $G \times_B \mathfrak{b}^k$ by the restriction $\gamma_{\mathcal{X}}$ of $\gamma_{\rm n}^{(k)}$ to $G \times_B \mathfrak{b}^k$. Then $\mathcal{B}_{\mathcal{X}}^{(k)} \setminus W_k$ is contained in the image of $G \times_B (\mathfrak{b} \setminus \mathfrak{b}_{\rm reg})^k$ by $\gamma_{\mathcal{X}}$. As a result (see Lemma 8.1),

$$\dim \mathcal{B}_{\mathcal{X}}^{k} \setminus W_k \leq n + k(\mathrm{b}_{\mathfrak{g}} - 2)$$

whence the assertion. □

Proposition 3.10. (i) *The varieties $\mathcal{B}_{\rm n}^{(k)}$ and $\mathcal{B}_{\mathcal{X}}^{(k)}$ are equal. Moreover, $\gamma_{\rm n} = \gamma_{\mathcal{X}}$.*

(ii) *The restriction to $\mathrm{S}(\mathfrak{h})^{\otimes k}$ of the quotient morphism $\Bbbk[\mathcal{X}]^{\otimes k} \rightarrow \Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ is an embedding.*

(iii) *The algebra $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Moreover, η is the restriction of ϖ to $\mathcal{B}_{\mathcal{X}}^{(k)}$.*

(iv) *The restriction of $\gamma_{\mathcal{X}}$ to $\gamma_{\mathcal{X}}^{-1}(W_k)$ is an isomorphism onto W_k.*

Proof. (i) According to Corollary 3.6, from the short exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{G^k \times_{B^k} \mathfrak{b}^k} \longrightarrow \mathcal{O}_{G \times_B \mathfrak{b}^k} \longrightarrow 0$$

one deduces the short exact sequence

$$0 \longrightarrow \mathrm{H}^0(G^k \times_{B^k} \mathfrak{b}^k, \mathcal{J}) \longrightarrow \mathrm{H}^0(G^k \times_{B^k} \mathfrak{b}^k, \mathcal{O}_{G^k \times_{B^k} \mathfrak{b}^k}) \longrightarrow \mathrm{H}^0(G \times_B \mathfrak{b}^k, \mathcal{O}_{G \times_B \mathfrak{b}^k}) \longrightarrow 0$$

In particular, the restriction map

$$\mathrm{H}^0(G^k \times_{B^k} \mathfrak{b}^k, \mathcal{O}_{G^k \times_{B^k} \mathfrak{b}^k}) \longrightarrow \mathrm{H}^0(G \times_B \mathfrak{b}^k, \mathcal{O}_{G \times_B \mathfrak{b}^k})$$

is surjective. Since $\Bbbk[\mathcal{X}]$ equals $\mathrm{H}^0(G\times_B \mathfrak{b}, \mathcal{O}_{G\times_B\mathfrak{b}})$ by Lemma 3.1,(v), the image of this map equals $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ by Corollary 3.8,(i). Moreover, according to Lemma 1.1, $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] = \mathrm{H}^0(G\times_B \mathfrak{b}^k, \mathcal{O}_{G\times_B\mathfrak{b}^k})$ since $G\times_B \mathfrak{b}^k$ is a desingularization of the normal variety $\mathcal{B}_{\mathrm{n}}^{(k)}$ by Lemma 2.1,(i). Hence $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] = \Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ and $\gamma_{\mathrm{n}} = \gamma_{\mathcal{X}}$.

(ii) According to (i), $\iota_{\mathrm{n}}(\mathfrak{b}^k)$ is a closed subvariety of $\mathcal{B}_{\mathrm{n}}^{(k)}$ and for p in $\mathrm{S}(\mathfrak{h})^{\otimes k}$, the restriction to $\iota_{\mathrm{n}}(\mathfrak{b}^k)$ of its image in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is the function

$$(x_1,\ldots,x_k,\overline{x_1},\ldots,\overline{x_k}) \longmapsto p(\overline{x_1},\ldots,\overline{x_k})$$

Hence the restriction to $\mathrm{S}(\mathfrak{h})^{\otimes k}$ of the quotient map $\Bbbk[\mathcal{X}]^{\otimes k} \rightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is an embedding.

(iii) The comorphism of the restriction of ϖ to $\mathcal{B}_{\mathcal{X}}^{(k)}$ is the embedding of to $\Bbbk[\mathcal{B}^{(k)}]$ into $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ so that η is the restriction of ϖ to $\mathcal{B}_{\mathcal{X}}^{(k)}$ by (i). Since $\Bbbk[\mathcal{X}^k]$ is generated by $\mathrm{S}(\mathfrak{g})^{\otimes k}$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$ and since the image of $\mathrm{S}(\mathfrak{g})^{\otimes k}$ by the quotient morphism equals $\Bbbk[\mathcal{B}^{(k)}]$, $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$.

(iv) Since the subset of Borel subalgebras containing a regular element is finite, the fibers of $\gamma_{\mathcal{X}}$ over the elements of W_k are finite. Indeed, according to Zariski Main Theorem [Mu88, §9], they have only one element since $\mathcal{B}_{\mathcal{X}}^{(k)}$ is normal by (i) and since $\gamma_{\mathcal{X}}$ is projective and birational. So, the restriction of $\gamma_{\mathcal{X}}$ to $\gamma_{\mathcal{X}}^{-1}(W_k)$ is a bijection onto the open subset W_k, whence the assertion by Zariski Main Theorem [Mu88, §9]. □

Remark 3.11. By Proposition 3.10,(i) and (iii), $\mathcal{B}_{\mathrm{n}}^{(k)}$ identifies with $\mathcal{B}_{\mathcal{X}}^{(k)}$ and η identifies with the restriction of ϖ to $\mathcal{B}_{\mathcal{X}}^{(k)}$ so that $\gamma_{\mathrm{n}} = \gamma_{\mathcal{X}}$ and $\iota_{\mathrm{n}} = \iota_1$.

Let consider on $\mathfrak{h}^k$ the diagonal action of $W(\mathcal{R})$.

Corollary 3.12. (i) *The subalgebra* $\mathrm{S}(\mathfrak{h})^{\otimes k}$ *of* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ *equals* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$.

(ii) *The subalgebras* $\Bbbk[\mathcal{B}^{(k)}]^G$ *and* $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ *of* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$ *are equal.*

Proof. (i) Let p be in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$ such that its restriction to $\iota_{\mathrm{n}}(\mathfrak{h}^k)$ equals 0. Since

$$\lim_{t\rightarrow 0} h(t).(x_1,\ldots,x_k) = (\overline{x_1},\ldots,\overline{x_k})$$

for all $(x_1,\ldots,x_k)$ in $\mathfrak{b}^k$, the restriction of p to $\iota_{\mathrm{n}}(\mathfrak{b}^k)$ equals 0 and $p=0$ since $\mathcal{B}_{\mathrm{n}}^{(k)} = G.\iota_{\mathrm{n}}(\mathfrak{b}^k)$.

According to Lemma 3.1,(v), $\mathrm{S}(\mathfrak{h})^{\otimes k} = (\Bbbk[\mathcal{X}]^{\otimes k})^{G^k}$. Hence $\mathrm{S}(\mathfrak{h})^{\otimes k}$ is a subalgebra of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$ since $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is a G-equivariant quotient of $\Bbbk[\mathcal{X}]^{\otimes k}$. For p in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$, let denote by $\overline{p}$ the element of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ such that

$$\overline{p}(x_1,\ldots,x_k) := p(x_1,\ldots,x_k,x_1,\ldots,x_k)$$

Then the restriction of $p-\overline{p}$ to $\iota_{\mathrm{n}}(\mathfrak{h}^k)$ equals 0. Moreover, if p is in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$, $p-\overline{p}$ is G-invariant. Hence $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G = \mathrm{S}(\mathfrak{h})^{\otimes k}$.

(ii) According to (i), the restriction from $\mathcal{B}^{(k)}$ to $\mathfrak{h}^k$ induces an embedding of $\Bbbk[\mathcal{B}^{(k)}]^G$ into $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$. Moreover, since G is reductive, $\Bbbk[\mathcal{B}^{(k)}]^G$ is the image of $(\mathrm{S}(\mathfrak{g})^{\otimes k})^G$ by the restriction morphism. According to [J07, Theorem 2.9 and some remark], the restriction morphism $(\mathrm{S}(\mathfrak{g})^{\otimes k})^G \rightarrow (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is surjective. Hence the restriction morphism $\Bbbk[\mathcal{B}^{(k)}]^G \rightarrow (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is surjective. Then the injection $\Bbbk[\mathcal{B}^{(k)}]^G \rightarrow (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is bijective since $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$ are graded quotients of $\mathrm{S}(\mathfrak{g})^{\otimes k}$. □

3.4. The natural action of $\Bbbk^*$ in $\mathfrak{g}^k$ induces an action of $\Bbbk^*$ on $\mathfrak{h}^k$, $\mathfrak{b}^k$, $\mathcal{B}^{(k)}$, $\mathcal{B}_{\mathrm{n}}^{(k)}$ and $G^k\times_{B^k}\mathfrak{b}^k$. In particular, $\Bbbk[\mathcal{B}^{(k)}]$ is a graded subalgebra of the graded algebra $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$.

Proposition 3.13. *The variety $\mathcal{B}_{\mathrm{n}}^{(k)}$ has rational singularities.*

Proof. From the short exact sequence of $\mathcal{O}_{G^k\times_{B^k}\mathfrak{b}^k}$-modules

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{G^k\times_{B^k}\mathfrak{b}^k} \longrightarrow \mathcal{O}_{G\times_B\mathfrak{b}^k} \longrightarrow 0$$

one deduces the cohomology long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(G^k\times_{B^k}\mathfrak{b}^k, \mathcal{O}_{G^k\times_{B^k}\mathfrak{b}^k}) \longrightarrow \mathrm{H}^i(G\times_B\mathfrak{b}^k, \mathcal{O}_{G\times_B\mathfrak{b}^k}) \longrightarrow \mathrm{H}^{i+1}(G^k\times_{B^k}\mathfrak{b}^k, \mathcal{J}) \longrightarrow \cdots$$

By Borel-Weil-Bott's Theorem [Dem68], for $i>0$, the first term equals 0 and by Corollary 3.6, the third term equals 0. Hence $\mathrm{H}^i(G\times_B\mathfrak{b}^k, \mathcal{O}_{G\times_B\mathfrak{b}^k}) = 0$ for all $i>0$, whence the proposition since $(G\times_B\mathfrak{b}^k, \gamma_{\mathrm{n}})$ is a desingularization of $\mathcal{B}_{\mathrm{n}}^{(k)}$ by Lemma 2.1,(i). □

Corollary 3.14. *Let M be a graded complement of $\Bbbk[\mathcal{B}^{(k)}]_+^G\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$.*

(i) *The algebra $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is a free extension of $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Moreover, M contains a basis of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{\otimes k}]$ over* $\mathrm{S}(\mathfrak{h})^{\otimes k}$.

(ii) *The intersection of M and $\mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is different from* 0.

Proof. (i) Let recall that $\mathcal{N}^{(k)}$ is the subset of elements $(x_1,\ldots,x_k)$ of $\mathcal{B}^{(k)}$ such that $x_1,\ldots,x_k$ are nilpotent and let recall that η is the canonical morphism from $\mathcal{B}_{\mathrm{n}}^{(k)}$ to $\mathcal{B}^{(k)}$. Let denote by τ the morphism from $\mathcal{B}_{\mathrm{n}}^{(k)}$ to $\mathfrak{h}^{\otimes k}$ whose comorphism is the injection of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$. First of all, $\mathcal{B}_{\mathrm{n}}^{(k)}$, $\mathfrak{h}^{\otimes k}$ and $\mathcal{N}^{(k)}$ have dimension $k\mathrm{b}_{\mathfrak{g}}+n$, $k\ell$, $(k+1)n$ respectively. Moreover, $\eta^{-1}(\mathcal{N}^{(k)})$ is the nullvariety of $\mathrm{S}_+(\mathfrak{h}^k)$ in $\mathcal{B}_{\mathrm{n}}^{(k)}$. In particular, the fiber at $(0,\ldots,0)$ of τ has minimal dimension. Since τ is an equivariant morphism with respect to the actions of $\Bbbk^*$ and since $(0,\ldots,0)$ is in the closure of all orbit of $\Bbbk^*$ in $\mathfrak{h}^k$, τ is an equidimensional morphism of dimension $\dim\mathcal{B}_{\mathrm{n}}^{(k)} - \dim\mathfrak{h}^{\otimes k}$. According to Proposition 3.13 and [El78], $\mathcal{B}_{\mathrm{n}}^{(k)}$ is Cohen-Macaulay. Then, by [MA86, Theorem 23.1], $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is a flat extension of $\mathrm{S}(\mathfrak{h})^{\otimes k}$.

The action of $\Bbbk^*$ on $\mathcal{B}_{\mathrm{n}}^{(k)}$ induces a $\mathbb{N}$-gradation of the algebra $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ compatible with the gradations of $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$ since τ is equivariant. Since M is a graded complement of $\Bbbk[\mathcal{B}^{(k)}]_+^G\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$, by induction on l,

$$\Bbbk[\mathcal{B}^{(k)}] = M\Bbbk[\mathcal{B}^{(k)}]^G + (\Bbbk[\mathcal{B}^{(k)}]_+^G)^l\Bbbk[\mathcal{B}^{(k)}]$$

Hence $\Bbbk[\mathcal{B}^{(k)}] = M\Bbbk[\mathcal{B}^{(k)}]^G$ since $\Bbbk[\mathcal{B}^{(k)}]$ is graded. Then, by Proposition 3.10,(iii) and Corollary 3.12,(ii), $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] = M\mathrm{S}(\mathfrak{h})^{\otimes k}$. In particular,

$$\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] = M + \mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$$

Then M contains a graded complement M' of $\mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$. Arguing as before, $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] = M'\mathrm{S}(\mathfrak{h})^{\otimes k}$ since $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is graded. By flatness, from the short exact sequence

$$0 \longrightarrow \mathrm{S}_+(\mathfrak{h}^k) \longrightarrow \mathrm{S}(\mathfrak{h})^{\otimes k} \longrightarrow \Bbbk \longrightarrow 0$$

one deduces the short exact sequence

$$0 \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]\otimes_{\mathrm{S}(\mathfrak{h})^{\otimes k}}\mathrm{S}_+(\mathfrak{h}^k) \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]\otimes_{\mathrm{S}(\mathfrak{h})^{\otimes k}}\Bbbk \longrightarrow 0$$

As a result, the canonical map $M'\otimes_{\Bbbk}\mathrm{S}(\mathfrak{h})^{\otimes k} \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is injective. Hence all basis of M' is a basis of the $\mathrm{S}(\mathfrak{h})^{\otimes k}$-module $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$, whence the assertion.

(ii) Let suppose that $M' = M$. One expects a contradiction. According to (i), the canonical maps

$$M \otimes_{\Bbbk} \mathrm{S}(\mathfrak{h})^{\otimes k} \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] \qquad M \otimes_{\Bbbk} \Bbbk[\mathcal{B}^{(k)}]^G \longrightarrow \Bbbk[\mathcal{B}^{(k)}]$$

are isomorphisms. Then, according to Lemma 1.2, there exists a group action of $W(\mathcal{R})$ on $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ extending the diagonal action of $W(\mathcal{R})$ in $\mathrm{S}(\mathfrak{h})^{\otimes k}$ and such that $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^{W(\mathcal{R})} = \Bbbk[\mathcal{B}^{(k)}]$ since $\Bbbk[\mathcal{B}^{(k)}] \cap \mathrm{S}(\mathfrak{h})^{\otimes k} = (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ by Corollary 3.12,(ii). Moreover, since $W(\mathcal{R})$ is finite, the subfield of invariant elements of the fraction field of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is the fraction field of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^{W(\mathcal{R})}$. Hence the action of $W(\mathcal{R})$ in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is trivial since $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ and $\Bbbk[\mathcal{B}^{(k)}]$ have the same fraction field, whence the contradiction since $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is strictly contained in $\mathrm{S}(\mathfrak{h})^{\otimes k}$. □

4. On the nullcone.

Let $k \geq 2$ be an integer. Let I be the ideal of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ generated by $\mathrm{S}_+(\mathfrak{h}^k)$ and let N be the subscheme of $\mathcal{B}_{\mathrm{n}}^{(k)}$ defined by I.

Lemma 4.1. *Let set* $\mathcal{N}_{\mathcal{X}}^{(k)} := \eta^{-1}(\mathcal{N}^{(k)})$.
(i) *The variety* $\mathcal{N}_{\mathcal{X}}^{(k)}$ *equals* $\gamma_{\mathrm{n}}(G \times_B \mathfrak{u}^k)$.
(ii) *The nullvariety of* I *in* $\mathcal{B}_{\mathrm{n}}^{(k)}$ *equals* $\mathcal{N}_{\mathcal{X}}^{(k)}$.
(iii) *The scheme* N *is smooth in codimension* 1.

Proof. (i) By definition, $\gamma^{-1}(\mathcal{N}^{(k)}) = G \times_B \mathfrak{u}^k$. Then, since $\gamma = \gamma_{\mathrm{n}} \circ \eta$, $\mathcal{N}_{\mathcal{X}}^{(k)} = \gamma_{\mathrm{n}}(G \times_B \mathfrak{u}^k)$.
(ii) Let $\mathcal{V}_I$ be the nullvariety of I in $\mathcal{B}_{\mathrm{n}}^{(k)}$. According to Proposition 3.10,(ii), for $(g, x_1, \ldots, x_k)$ in $G \times \mathfrak{b}^k$, $\gamma_{\mathrm{n}}((g, x_1, \ldots, x_k))$ is a zero of I if and only if $x_1, \ldots, x_k$ are nilpotent, whence the assertion.
(iii) According to Lemma 3.9,(i), one has an isomorphism of varieties

$$U_- \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{(k-1)} \longrightarrow W'_k \qquad (g, x_1, \ldots, x_k) \longmapsto (g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k})$$

Let J be the ideal of $\Bbbk[\mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{(k-1)}]$ generated by the functions $(x_1, \ldots, x_k) \mapsto \langle v, x_i \rangle$, $i = 1, \ldots, k$, $v \in \mathfrak{h}$ and let N_0 be the subscheme of $\mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{(k-1)}$ defined by the ideal J. Then the above map induces an isomorphism of $U_- \times N_0$ onto the open subset $W'_k \cap N$ of N. For all x in $\mathfrak{u}_{\mathrm{reg}} \times \mathfrak{u}^{(k-1)}$, the tangent space of N_0 at x equals $\mathfrak{u}^k$. Hence N_0 is smooth and $W'_k \cap N$ is smooth. Then, since N is a subscheme of $\mathcal{B}_{\mathrm{n}}^{(k)}$ invariant under the actions of G and $\mathrm{GL}_k(\Bbbk)$, the open subset $W_k \cap N$ of N is smooth by Lemma 3.9,(ii). By definition, $W_k \cap N = \eta^{-1}(V_k)$, whence the assertion by Corollary 2.3,(i) since η is finite. □

Proposition 4.2. *The variety* $\mathcal{N}_{\mathcal{X}}^{(k)}$ *is a normal variety and* I *is its ideal of definition in* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$. *In particular,* I *is prime.*

Proof. According to Corollary 3.14,(i), $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is a flat extension of $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Since $\mathcal{B}_{\mathrm{n}}^{(k)}$ is Cohen Macaulay, N is Cohen Macaulay by [MA86, Corollary of Theorem 23.2]. According to Lemma 4.1,(iii), N is smooth in codimension 1. Hence N is a normal scheme by Serre's normality criterion [Bou98, §1, no 10, Théorème 4]. According to Lemma 4.1,(ii), $\mathcal{N}_{\mathcal{X}}^{(k)}$ is the nullvariety of I in $\mathcal{B}_{\mathrm{n}}^{(k)}$. Moreover, $\mathcal{N}_{\mathcal{X}}^{(k)}$ is irreducible as image of the irreducible variety $G \times_B \mathfrak{u}^k$ by Lemma 4.1,(i). Hence I is prime and $\mathcal{N}_{\mathcal{X}}^{(k)}$ is a normal variety. □

Theorem 4.3. *Let I_0 be the ideal of $\Bbbk[\mathcal{B}^{(k)}]$ generated by $\Bbbk[\mathcal{B}^{(k)}]_+^G$.*

(i) *The ideal I_0 is strictly contained in the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$.*

(ii) *The nullcone $\mathcal{N}^{(k)}$ has rational singularities.*

Proof. (i) Since $\Bbbk[\mathcal{B}^{(k)}]_+^G$ is contained in $\mathrm{S}_+(\mathfrak{h}^k)$, I_0 is contained in $I \cap \Bbbk[\mathcal{B}^{(k)}]$. According to Lemma 4.1,(ii) and Proposition 4.2, $I \cap \Bbbk[\mathcal{B}^{(k)}]$ is the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$. Let M be a graded complement of $\Bbbk[\mathcal{B}^{(k)}]_+^G \Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$. According to Corollary 3.14,(ii), $I \cap M$ is different from 0. Hence I_0 is strictly contained in $I \cap \Bbbk[\mathcal{B}^{(k)}]$, whence the assertion.

(ii) According to Proposition 3.10,(iii) and Proposition 4.2, the restriction to $\Bbbk[\mathcal{B}^{(k)}]$ of the quotient map from $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ to $\Bbbk[\mathcal{N}_{\mathcal{X}}^{(k)}]$ is surjective. Furthermore, the image of $\Bbbk[\mathcal{B}^{(k)}]$ by this morphism equals $\Bbbk[\mathcal{N}^{(k)}]$ since $\mathcal{N}_{\mathcal{X}}^{(k)} = \eta^{-1}(\mathcal{N}^{(k)})$, whence $\Bbbk[\mathcal{N}^{(k)}] = \Bbbk[\mathcal{N}_{\mathcal{X}}^{(k)}]$. As a result, $\mathcal{N}^{(k)}$ has rational singularities since $\mathcal{N}_{\mathcal{X}}^{(k)}$ is normal and since the normalization of $\mathcal{N}^{(k)}$ has rational singularities by Theorem 2.4. □

5. Main varieties.

Let denote by X the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under B. According to Lemma 1.4, $G.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under G.

5.1. For α in $\mathcal{R}$, let denote by $\mathfrak{h}_\alpha$ the kernel of α. Let set $V_\alpha := \mathfrak{h}_\alpha \oplus \mathfrak{g}^\alpha$ and let denote by X_α the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of V_α under B.

Lemma 5.1. *Let α be in $\mathcal{R}_+$. Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$ and let P be its normalizer in G.*

(i) *The subset $P.X$ of $\mathrm{Gr}_\ell(\mathfrak{g})$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under P.*

(ii) *The closed set X_α of $\mathrm{Gr}_\ell(\mathfrak{g})$ is an irreducible component of $X \setminus B.\mathfrak{h}$.*

(iii) *The set $P.X_\alpha$ is an irreducible component of $P.X \setminus P.\mathfrak{h}$.*

(iv) *The varieties $X \setminus B.\mathfrak{h}$ and $P.X \setminus P.\mathfrak{h}$ are equidimenional of codimension 1 in X and $P.X$ respectively.*

Proof. (i) Since X is a B-invariant closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$, $P.X$ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$ by Lemma 1.4. Hence $\overline{P.\mathfrak{h}}$ is contained in $P.X$ since $\mathfrak{h}$ is in X, whence the assertion since $\overline{P.\mathfrak{h}}$ is a P-invariant subset containing X.

(ii) Denoting by H_α the coroot of α,

$$\lim_{t\to\infty} \exp(t \mathrm{ad}\, x_\alpha)(\frac{-1}{2t} H_\alpha) = x_\alpha$$

So V_α is in the closure of the orbit of $\mathfrak{h}$ under the one parameter subgroup of G generated by $\mathrm{ad}\, x_\alpha$. As a result, X_α is a closed subset of $X \setminus B.\mathfrak{h}$ since V_α is not a Cartan subalgebra. Moreover, X_α has dimension $n-1$ since the normalizer of V_α in $\mathfrak{g}$ is $\mathfrak{h} + \mathfrak{g}^\alpha$. Hence X_α is an irreducible component of $X \setminus B.\mathfrak{h}$ since X has dimension n.

(iii) Since X_α is a B-invariant closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$, $P.X_\alpha$ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$ by Lemma 1.4. According to (ii), $P.X_\alpha$ is contained in $P.X \setminus P.\mathfrak{h}$ and it has dimension $\dim \mathfrak{p} - \ell - 1$, whence the assertion since $P.X$ has dimension $\dim \mathfrak{p} - \ell$.

(iv) Let P_{u} be the unipotent radical of P and let L be the reductive factor of P whose Lie algebra contains $\mathrm{ad}\,\mathfrak{h}$. Let denote by $N_L(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in L. Since $B.\mathfrak{h}$ and $P.\mathfrak{h}$ are isomorphic to U and

$L/N_L(\mathfrak{h})\times P_{\mathrm{u}}$ respectively, they are affine open subsets of X and $P.X$ respectively, whence the assertion by [EGAIV, Corollaire 21.12.7]. □

For x in V, let set:

$$V_x := \mathrm{span}(\{\varepsilon_1(x),\ldots,\varepsilon_\ell(x)\})$$

Lemma 5.2. *Let Δ be the set of elements (x,V) of $\mathfrak{g}\times G.X$ such that x is in V.*

(i) *For (x,V) in $\mathfrak{b}\times X$, (x,V) is in the closure of $B.(\mathfrak{h}_{\mathrm{reg}}\times\{\mathfrak{h}\})$ in $\mathfrak{b}\times\mathrm{Gr}_\ell(\mathfrak{b})$ if and only if x is in V.*

(ii) *The set Δ is the closure in $\mathfrak{g}\times\mathrm{Gr}_\ell(\mathfrak{g})$ of $G.(\mathfrak{h}_{\mathrm{reg}}\times\{\mathfrak{h}\})$.*

(iii) *For (x,V) in Δ, V_x is contained in V.*

Proof. (i) Let Δ' be the subset of elements (x,V) of $\mathfrak{b}\times X$ such that x is in V and let Δ'_0 be the closure of $B.(\mathfrak{h}_{\mathrm{reg}}\times\{\mathfrak{h}\})$ in $\mathfrak{b}\times\mathrm{Gr}_\ell(\mathfrak{b})$. Then Δ' is a closed subset of $\mathfrak{b}\times\mathrm{Gr}_\ell(\mathfrak{b})$ containing Δ'_0. Let (x,V) be in Δ'. Let E be a complement of V in $\mathfrak{b}$ and let Ω_E be the set of complements of E in $\mathfrak{g}$. Then Ω_E is an open neighborhood of V in $\mathrm{Gr}_\ell(\mathfrak{b})$. Moreover, the map

$$\mathrm{Hom}_{\Bbbk}(V,E)\stackrel{\kappa}{\longrightarrow}\Omega_E \qquad \varphi\longmapsto\kappa(\varphi):=\mathrm{span}(\{v+\varphi(v)\mid v\in V\})$$

is an isomorphism of varieties. Let Ω_E^c be the inverse image of the set of Cartan subalgebras. Then 0 is in the closure of Ω_E^c in $\mathrm{Hom}_{\Bbbk}(V,E)$ since V is in X. For all φ in Ω_E^c, $(x+\varphi(x),\kappa(\varphi))$ is in Δ'_0. Hence (x,V) is in Δ'_0.

(ii) Let (x,V) be in Δ. For some g in G, $g(V)$ is in X. So by (i), $(g(x),g(V))$ is in Δ'_0 and (x,V) is in the closure of $G.(\mathfrak{h}_{\mathrm{reg}}\times\{\mathfrak{h}\})$ in $\mathfrak{g}\times\mathrm{Gr}_\ell(\mathfrak{g})$, whence the assertion.

(iii) For $i=1,\ldots,\ell$, let Δ_i be the set of elements (x,V) of Δ such that $\varepsilon_i(x)$ is in V. Then Δ_i is a closed subset of $\mathfrak{g}\times G.X$, invariant under the action of G in $\mathfrak{g}\times\mathrm{Gr}_\ell(\mathfrak{g})$ since ε_i is a G-equivariant map. For all (g,x) in $G\times\mathfrak{h}_{\mathrm{reg}}$, $(g(x),g(\mathfrak{h}))$ is in Δ_i since $\varepsilon_i(g(x))$ centralizes $g(x)$. Hence $\Delta_i=\Delta$ since $G.(\mathfrak{h}_{\mathrm{reg}}\times\{\mathfrak{h}\})$ is dense in Δ by (ii). As a result, for all V in $G.X$ and for all x in V, $\varepsilon_1(x),\ldots,\varepsilon_\ell(x)$ are in V. □

Corollary 5.3. *Let (x,V) be in Δ and let $\mathfrak{z}$ be the centre of $\mathfrak{g}^{x_{\mathrm{s}}}$.*

(i) *The subspace $\mathfrak{z}$ is contained in $V_{x_{\mathrm{s}}}$ and V.*

(ii) *The space V is an algebraic, commutative subalgebra of $\mathfrak{g}$.*

Proof. (i) If x is regular semisimple, V is a Cartan subalgebra of $\mathfrak{g}$. Let suppose that x is not regular semisimple. Let denote by $\mathfrak{z}$ be the centre of $\mathfrak{g}^{x_{\mathrm{s}}}$. Let $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$ be the nilpotent cone of $\mathfrak{g}^{x_s}$ and let Ω_{reg} be the regular nilpotent orbit of $\mathfrak{g}^{x_{\mathrm{s}}}$. For all y in Ω_{reg}, $x_{\mathrm{s}}+y$ is in $\mathfrak{g}_{\mathrm{reg}}$ and $\varepsilon_1(x_{\mathrm{s}}+y),\ldots,\varepsilon_\ell(x_{\mathrm{s}}+y)$ is a basis of $\mathfrak{g}^{x_{\mathrm{s}}+y}$ by [Ko63, Theorem 9]. Then for all z in $\mathfrak{z}$, there exist regular functions on Ω_{reg}, $a_{1,z},\ldots,a_{\ell,z}$, such that

$$z=a_{1,z}(y)\varepsilon_1(x_{\mathrm{s}}+y)+\cdots+a_{\ell,z}(y)\varepsilon_\ell(x_{\mathrm{s}}+y)$$

for all y in Ω_{reg}. Furthermore, these functions are uniquely defined by this equality. Since $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$ is a normal variety and since $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}\setminus\Omega_{\mathrm{reg}}$ has codimension 2 in $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$, the functions $a_{1,z},\ldots,a_{\ell,z}$ have regular extensions to $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$. Denoting again by $a_{i,z}$ the regular extension of $a_{i,z}$ for $i=1,\ldots,\ell$,

$$z=a_{1,z}(y)\varepsilon_1(x_{\mathrm{s}}+y)+\cdots+a_{\ell,z}(y)\varepsilon_\ell(x_{\mathrm{s}}+y)$$

for all y in $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$. As a result, $\mathfrak{z}$ is contained in V_x. Hence $\mathfrak{z}$ is contained in V by Lemma 5.2,(iii).

(ii) Since the set of commutative subalgebras of dimension ℓ is closed in $\mathrm{Gr}_\ell(\mathfrak{g})$, V is a commutative subalgebra of $\mathfrak{g}$. According to (i), the semisimple and nilpotent components of the elements of V are contained in V. For x in $V \setminus \mathfrak{N}_{\mathfrak{g}}$, all the replica of $x_{\rm s}$ are contained in $\mathfrak{z}$. Hence V is an algebraic subalgebra of $\mathfrak{g}$ by (i). □

5.2. For s in $\mathfrak{h}$, let denote by X^s the subset of elements of X, contained in $\mathfrak{g}^s$.

Lemma 5.4. *Let s be in $\mathfrak{h}$ and let $\mathfrak{z}$ be the centre of $\mathfrak{g}^s$.*

(i) *The set X^s is the closure in $\mathrm{Gr}_\ell(\mathfrak{g}^s)$ of the orbit of $\mathfrak{h}$ under B^s.*

(ii) *The set of elements of $G.X$ containing $\mathfrak{z}$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under G^s.*

Proof. (i) Let set $\mathfrak{p} := \mathfrak{g}^s + \mathfrak{b}$, let P be the normalizer of $\mathfrak{p}$ in G and let $\mathfrak{p}_{\rm u}$ be the nilpotent radical of $\mathfrak{p}$. For g in P, let denote by $\overline{g}$ its image by the canonical projection from P to G^s. Let Z be the closure in $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathrm{Gr}_\ell(\mathfrak{g})$ of the image of the map

$$B \longrightarrow \mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b}) \qquad g \longmapsto (g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$$

and let Z' be the subset of elements (V, V') of $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b})$ such that

$$V' \subset \mathfrak{g}^{\rm s} \cap \mathfrak{b} \text{ and } V \subset V' \oplus \mathfrak{p}_{\rm u}$$

Then Z' is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b})$ and Z is contained in Z' since $(g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$ is in Z' for all g in B. Since $\mathrm{Gr}_\ell(\mathfrak{b})$ is a projective variety, the images of Z by the projections $(V, V') \mapsto V$ and $(V, V') \mapsto V'$ are closed in $\mathrm{Gr}_\ell(\mathfrak{b})$ and they equal X and $\overline{B^s.\mathfrak{h}}$ respectively. Furthermore, $\overline{B^s.\mathfrak{h}}$ is contained in X^s.

Let V be in X^s. For some V' in $\mathrm{Gr}_\ell(\mathfrak{b})$, (V, V') is in Z. Since

$$V \subset \mathfrak{g}^s, \; V' \subset \mathfrak{g}^s, \; V \subset V' \oplus \mathfrak{p}_{\rm u}$$

$V = V'$ so that V is in $\overline{B^s.\mathfrak{h}}$, whence the assertion.

(ii) Since $\mathfrak{z}$ is contained in $\mathfrak{h}$, all element of $\overline{G^s.\mathfrak{h}}$ is an element of $G.X$ containing $\mathfrak{z}$. Let V be in $G.X$, containing $\mathfrak{z}$. Since V is a commutative subalgebra of $\mathfrak{g}^s$ and since $\mathfrak{g}^s \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}^s$, for some g in G^s, $g(V)$ is contained in $\mathfrak{b} \cap \mathfrak{g}^s$. So, one can suppose that V is contained in $\mathfrak{b}$. According to the Bruhat decomposition of G, since X is B-invariant, for some b in U and for some w in $W(\mathcal{R})$, V is in $bw.X$. Let set:

$$\mathcal{R}_{+,w} := \{\alpha \in \mathcal{R}_+ \mid w(\alpha) \in \mathcal{R}_+\} \qquad \mathcal{R}'_{+,w} := \{\alpha \in \mathcal{R}_+ \mid w(\alpha) \notin \mathcal{R}_+\}$$

$$\mathfrak{u}_1 := \bigoplus_{\alpha\in\mathcal{R}_{+,w}} \mathfrak{g}^{w(\alpha)} \qquad \mathfrak{u}_2 := \bigoplus_{\alpha\in-\mathcal{R}'_{+,w}} \mathfrak{g}^{w(\alpha)} \qquad \mathfrak{u}_3 := \bigoplus_{\alpha\in\mathcal{R}'_{+,w}} \mathfrak{g}^{w(\alpha)}$$

$$B^w := wBw^{-1} \qquad \mathfrak{b}^w := \mathfrak{h} \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_3$$

so that $\mathrm{ad}\, \mathfrak{b}^w$ is the Lie algebra of B^w and $w.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under B^w. Moreover, $\mathfrak{u}$ is the direct sum of $\mathfrak{u}_1$ and $\mathfrak{u}_2$. For $i = 1, 2$, let denote by U_i the closed subgroup of U whose Lie algebra is $\mathrm{ad}\, \mathfrak{u}_i$. Then $U = U_2U_1$ and $b = b_2b_1$ with b_i in U_i for $i = 1, 2$. Since $w^{-1}(\mathfrak{u}_1)$ is contained in $\mathfrak{u}$ and since X is invariant under B, $b_2b_1w.X = b_2w.X$. Since $b_2^{-1}(V)$ is in $w.X$ and since V is contained in $\mathfrak{b}$,

$$b_2^{-1}(V) \subset \mathfrak{b} \cap \mathfrak{b}^w = \mathfrak{h} \oplus \mathfrak{u}_1$$

Let set:

$$\mathfrak{u}_{2,1} := \mathfrak{u}_2 \cap \mathfrak{g}^s \qquad \mathfrak{u}_{2,2} := \mathfrak{u}_2 \cap \mathfrak{p}_{\rm u}$$

and for $i = 1,2$, let denote by $U_{2,i}$ the closed subgroup of U_2 whose Lie algebra is $\mathrm{ad}\,\mathfrak{u}_{2,i}$. Then $\mathfrak{u}_2$ is the direct sum of $\mathfrak{u}_{2,1}$ and $\mathfrak{u}_{2,2}$ and $U_2 = U_{2,1}U_{2,2}$ so that $b_2 = b_{2,1}b_{2,2}$ with $b_{2,i}$ in $U_{2,i}$ for $i = 1,2$. As a result, $\mathfrak{z}$ is contained in $b_{2,1}^{-1}(V)$ and $b_{2,2}^{-1}(\mathfrak{z})$ is contained $\mathfrak{h} \oplus \mathfrak{u}_1$. Hence $b_{2,2}^{-1}(\mathfrak{z}) = \mathfrak{z}$ since $\mathfrak{u}_1 \cap \mathfrak{u}_{2,2} = \{0\}$.

Let suppose $b_{2,2} \neq 1$. One expects a contradiction. For some x in $\mathfrak{u}_{2,2}$, $b_{2,2} = \exp(\mathrm{ad}\,x)$. The space $\mathfrak{u}_{2,2}$ is a direct sum of root spaces since $\mathfrak{u}_2$ and $\mathfrak{p}_{\mathrm{u}}$ are too. Let $\alpha_1,\dots,\alpha_m$ be the positive roots such that the corresponding root spaces are contained in $\mathfrak{u}_{2,2}$. They are ordered so that for $i \leq j$, $\alpha_j - \alpha_i$ is a positive root if it is a root. For $i = 1,\dots,m$, let c_i be the coordinate of x at x_{α_i} and let i_0 be the smallest integer such that $c_{i_0} \neq 0$. For all z in $\mathfrak{z}$,

$$b_{2,2}^{-1}(z) - z - c_{i_0}\alpha_{i_0}(z)x_{\alpha_{i_0}} \in \bigoplus_{j>i_0} \mathfrak{g}^{\alpha_j}$$

whence the contradiction since for some z in $\mathfrak{z}$, $\alpha_{i_0}(z) \neq 0$. As a result, $b_{2,1}^{-1}(V)$ is an element of $w.X = \overline{B^w.\mathfrak{h}}$, contained in $\mathfrak{g}^s$. So, by (i), $b_{2,1}^{-1}(V)$ and V are in $\overline{G^s.\mathfrak{h}}$, whence the assertion. □

5.3. For x in $\mathfrak{g}$, let denote by R_x the subset of stabilizers of regular linear forms on $\mathfrak{g}^x$ under the coadjoint action. According to [Y06a] and [deG08], all element of R_x is a commutative subalgebra of dimension ℓ of $\mathfrak{g}$. For x and v in $\mathfrak{g}$, the stabilizer of the linear form $w \mapsto \langle v,w\rangle$ on $\mathfrak{g}^x$ under the coadjoint action is denoted by $(\mathfrak{g}^x)^v$.

Lemma 5.5. *Let x be in $\mathfrak{g}$.*

(i) *For all v in $\mathfrak{g}$, there exists a positive integer d and a regular map β_v from $\mathbb{P}^1(\Bbbk)$ to $\mathrm{Gr}_d(\mathfrak{g})$ such that $\beta_v(t) = \mathfrak{g}^{tx+v}$ for all t in a dense open subset of $\Bbbk$. Moreover, $\beta_v(\infty)$ is contained in $(\mathfrak{g}^x)^v$.*

(ii) *For all v in a dense open subset Ω of $\mathfrak{g}$, $x+v$ is regular semisimple and the linear form $w \mapsto \langle v,w\rangle$ on $\mathfrak{g}^x$ is regular.*

Proof. (i) Let d be the minimal dimension of the $\mathfrak{g}^{tx+v}$'s, $t \in \Bbbk$. Then for all t in a dense open subset Ω_v of $\Bbbk$, the map

$$\Omega_v \longrightarrow \mathrm{Gr}_d(\mathfrak{g}) \qquad t \longmapsto \mathfrak{g}^{tx+v}$$

is regular, whence the assertion by [Sh94, Ch. VI, Theorem 1].

Let E be a complement of $\beta_v(\infty)$ in $\mathfrak{g}$ and let Ω_E be the set of complements of E in $\mathfrak{g}$. Then Ω_E is an open neighborhood of $\beta_v(\infty)$ in $\mathrm{Gr}_\ell(\mathfrak{g})$ and the map

$$\mathrm{Hom}_{\Bbbk}(\beta_v(\infty),E) \longrightarrow \Omega_E \qquad \varphi \longmapsto \mathrm{span}(\{w+\varphi(w) \mid v \in \beta_w(\infty)\})$$

is an isomorphism. Let denote by χ this isomorphism. For all t in a nonempty open subset T of $\Bbbk^*$, there exists a unique φ_t in $\mathrm{Hom}_{\Bbbk}(\beta_v(\infty),E)$ such that $\chi(\varphi_t) = \mathfrak{g}^{tx+v}$. Then

$$\lim_{t\longrightarrow\infty} \varphi_t = 0$$

and for all (w,t) in $\beta_v(\infty)\times T$, one has

$$0 = [w+\varphi_t(w), tx+v] = [w+\varphi_t(w), x+\frac{1}{t}v]$$

whence $\beta_v(\infty) \subset \mathfrak{g}^x$. Moreover, for all all w' in $\mathfrak{g}^x$,

$$0 = \langle w', [w+\varphi_t(w), tx+v]\rangle = -\langle w+\varphi_t(w), [w',v]\rangle$$

whence $\beta_v(\infty) \subset (\mathfrak{g}^x)^v$.

(ii) Since $\mathfrak{g}_{\mathrm{reg,ss}}$ is a dense open subset of $\mathfrak{g}$, for all v in a dense open subset, $x+v$ is regular semisimple. Since the map

$$\mathfrak{g} \longrightarrow (\mathfrak{g}^x)^* \qquad v \longmapsto (w \mapsto \langle v, w\rangle)$$

is a dominant morphism, for all v in a dense open subset of $\mathfrak{g}$, the linear form $w \mapsto \langle v, w\rangle$ on $\mathfrak{g}^x$ is regular, whence the assertion. □

Corollary 5.6. *For x in $\mathfrak{g}$, R_x is contained in $G.X$.*

Proof. Let v be in the open subset Ω of Lemma 5.5,(ii). Then $\mathfrak{g}^{tx+v}$ is a Cartan subalgebra of $\mathfrak{g}$ for all t in a dense open subset of $\Bbbk$. So $\beta_v(\infty)$ is in $G.X$ and by Lemma 5.5,(i), $\beta_v(\infty) = (\mathfrak{g}^x)^v$ since the index of $\mathfrak{g}^x$ equals ℓ. Denoting by $(\mathfrak{g}^x)^*_{\mathrm{reg}}$ the set of regular linear forms on $\mathfrak{g}^x$, the map

$$(\mathfrak{g}^x)^*_{\mathrm{reg}} \longrightarrow \mathrm{Gr}_\ell(\mathfrak{g}) \qquad v \longmapsto (\mathfrak{g}^x)^v$$

is regular. Hence R_x is contained in $G.X$ since the projection of Ω to $(\mathfrak{g}^x)^*$ is dense in $(\mathfrak{g}^x)^*_{\mathrm{reg}}$ and since $G.X$ is closed in $\mathrm{Gr}_\ell(\mathfrak{g})$. □

For E a subspace of $\mathfrak{g}$ of even dimension $2m$ and for $\underline{e} = e_1, \ldots, e_{2m}$ a basis of E, let set:

$$p_{E,\underline{e}} := \det \begin{bmatrix} [e_1, e_1] & \cdots & [e_1, e_{2m}] \\ \vdots & \ddots & \vdots \\ [e_{2m}, e_1] & \cdots & [e_{2m}, e_{2m}] \end{bmatrix}$$

The element $p_{E,\underline{e}}$ of $\mathrm{S}(\mathfrak{g})$, up to a multiplicative scalar, does not depend on the basis $\underline{e}$. So, when $p_{E,\underline{e}}$ is different from zero, one will say that p_E is different from zero. Otherwise, one will say $p_E = 0$.

Lemma 5.7. *Let x be in $\mathfrak{g}$.*

(i) *For V in $\mathrm{Gr}_\ell(\mathfrak{g}^x)$, V is in $\overline{R_x}$ if and only if for all complement E of V in $\mathfrak{g}^x$, p_E is different from zero.*

(ii) *For V in $\mathrm{Gr}_\ell(\mathfrak{g})$, V is in $G.X$ if and only if for all complement E of V in $\mathfrak{g}$, p_E is different from zero.*

(iii) *For E in $\mathrm{Gr}_{\dim \mathfrak{g}^x - \ell}(\mathfrak{g}^x)$ such that $p_E \neq 0$, $p_{E\oplus F} \neq 0$ for all complement F of $\mathfrak{g}^x$ in $\mathfrak{g}$.*

Proof. (i) and (ii) Let denote by $\mathfrak{a}$ the Lie algebra $\mathfrak{g}$ or $\mathfrak{g}^x$. For v in $\mathfrak{g}$, let denote by $\mathfrak{a}^v$ the stabilizer of the linear form $w \mapsto \langle v, w\rangle$ on $\mathfrak{a}$ and let set:

$$Z_0 := \begin{cases} G.X & \text{if} \quad \mathfrak{a} = \mathfrak{g} \\ \overline{R_x} & \text{if} \quad \mathfrak{a} = \mathfrak{g}^x \end{cases}$$

Let V be in $\mathrm{Gr}_\ell(\mathfrak{a})$. For all complement E of V in $\mathfrak{a}$, E has even dimension. Let suppose $p_E \neq 0$ for all complement E of V in $\mathfrak{a}$. Let $E_1, \ldots, E_m$ be some complements of V in $\mathfrak{a}$. Then for all v in a dense open subset of $\mathfrak{g}$, v is not a zero of $p_{E_1}, \ldots, p_{E_m}$ and the linear form $w \mapsto \langle v, w\rangle$ on $\mathfrak{a}$ is regular. Hence for $i = 1, \ldots, m$, $\mathfrak{a}^v$ is a complement of E_i in $\mathfrak{a}$. As a result, V is in Z_0. Conversely, let suppose that V is in Z_0 and let E be a complement of V in $\mathfrak{a}$. Then for some v in $\mathfrak{g}$, $\mathfrak{a}^v$ is a complement of E in $\mathfrak{a}$ so that v is not a zero of p_E.

(iii) Let v be in $\mathfrak{g}$ such that $p_E(v) \neq 0$ and such that the linear form $w \mapsto \langle v, w\rangle$ on $\mathfrak{g}^x$ is regular. Then $(\mathfrak{g}^x)^v$ is a complement of E in $\mathfrak{g}^x$. According to Lemma 5.5,(iii), $(\mathfrak{g}^x)^v$ is in $G.X$. For all complement F of $\mathfrak{g}^x$ in $\mathfrak{g}$, $E \oplus F$ is a complement of $(\mathfrak{g}^x)^v$ in $\mathfrak{g}$, whence the assertion by (ii). □

5.4. Let call a torus of $\mathfrak{g}$ a commutative algebraic subalgebra of $\mathfrak{g}$ whose all elements are semisimple. For x in $\mathfrak{g}$, let denote by Z_x the subset of elements of $G.X$ containing x and let denote by $(G^x)_0$ the identity component of G^x.

Lemma 5.8. *Let x be in $\mathfrak{N}_{\mathfrak{g}}$ and let Z be an irreducible component of Z_x. Let suppose that some element of Z is not contained in $\mathfrak{N}_{\mathfrak{g}}$.*

(i) *For some torus $\mathfrak{s}$ of $\mathfrak{g}^x$, all element of a dense open subset of Z contains a conjugate of $\mathfrak{s}$ under $(G^x)_0$.*

(ii) *For some s in $\mathfrak{s}$ and for some irreducible component Z_1 of Z_{s+x}, Z is the closure in* $\mathrm{Gr}_\ell(\mathfrak{g})$ *of $(G^x)_0.Z_1$.*

(iii) *If Z_1 has dimension smaller than* $\dim \mathfrak{g}^{s+x} - \ell$*, then Z has dimension smaller than* $\dim \mathfrak{g}^x - \ell$.

Proof. (i) After some conjugation by an element of G, one can suppose that $\mathfrak{g}^x \cap \mathfrak{b}$ and $\mathfrak{g}^x \cap \mathfrak{h}$ are a Borel subalgebra and a maximal torus of $\mathfrak{g}^x$ respectively. Let Z_0 be the subset of elements of Z contained in $\mathfrak{b}$ and let $(B^x)_0$ be the identity component of B^x. Since Z is an irreducible component of Z_x, Z is invariant under $(G^x)_0$ and $Z = (G^x)_0.Z_0$. Since $(G^x)_0/(B^x)_0$ is a projective variety, according to the proof of Lemma 1.4, $(G^x)_0.Z_*$ is a closed subset of Z for all closed subset Z_* of Z. Hence for some irreducible component Z_* of Z_0, $Z = (G^x)_0.Z_*$. According to Corollary 5.3,(ii), for all V in Z_*, there exists a torus $\mathfrak{s}$, contained in $\mathfrak{g}^x \cap \mathfrak{h}$ and verifying the following two conditions:

(1) V is contained in $\mathfrak{s} + (\mathfrak{g}^x \cap \mathfrak{u})$,
(2) V contains a conjugate of $\mathfrak{s}$ under $(B^x)_0$.

Let $\mathfrak{s}$ be a torus of maximal dimension verifying Conditions (1) and (2) for some V in Z_*. By hypothesis, $\mathfrak{s}$ has positive dimension. Let $Z_{\mathfrak{s}}$ be the subset of elements of Z_* verifying Conditions (1) and (2) with respect to $\mathfrak{s}$. By maximality of $\dim \mathfrak{s}$, for V in $Z_* \setminus Z_{\mathfrak{s}}$, $\dim V \cap \mathfrak{u} > \ell - \dim \mathfrak{s}$ or $\dim V \cap \mathfrak{u} = \ell - \dim \mathfrak{s}$ and V is contained in $\mathfrak{s}' + \mathfrak{u}$ for some torus of dimension $\dim \mathfrak{s}$, different from $\mathfrak{s}$. By rigidity of tori, $\mathfrak{s}$ is not in the closure in $\mathrm{Gr}_{\dim \mathfrak{s}}(\mathfrak{h})$ of the set of tori different from $\mathfrak{s}$. Hence $Z_* \setminus Z_{\mathfrak{s}}$ is a closed subset of Z_* since for all V in $\overline{Z_* \setminus Z_{\mathfrak{s}}}$, $\dim V \cap \mathfrak{u}$ has dimension at least $\ell - \dim \mathfrak{s}$. As a result, $(G^x)_0.Z_{\mathfrak{s}}$ contains a dense open subset whose all elements contain a conjugate of $\mathfrak{s}$ under $(G^x)_0$.

(ii) For some s in $\mathfrak{s}$, $\mathfrak{g}^s$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. Let Z^s be the subset of elements of Z containing s. Then Z^s is contained in Z_{s+x} and according to Corollary 5.3,(i), Z^s is the subset of elements of Z, containing $\mathfrak{s}$. By (i), for some irreducible component Z'_1 of Z^s, $(G^x)_0.Z'_1$ is dense in Z. Let Z_1 be an irreducible component of Z_{s+x}, containing Z'_1. According to Corollary 5.3,(ii), Z_1 is contained in Z_x since x is the nilpotent component of $s + x$. So $Z_1 = Z'_1$ and $(G^x)_0.Z_1$ is dense in Z.

(iii) Since Z_1 is an irreducible component of Z_{s+x}, Z_1 is invariant under the identity component of G^{s+x}. Moreover, G^{s+x} is contained in G^x since x is the nilpotent component of $s + x$. As a result, by (ii),

$$\dim Z \leq \dim \mathfrak{g}^x - \dim \mathfrak{g}^{s+x} + \dim Z_1$$

whence the assertion. □

Let denote by C_h the G-invariant closed cone generated by h.

Lemma 5.9. *Let suppose $\mathfrak{g}$ semisimple. Let Γ be the closure in $\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g})$ of the image of the map*

$$\Bbbk^* \times G \longrightarrow \mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g}) \qquad (t, g) \longmapsto (tg(h), g(\mathfrak{h}))$$

and let Γ_0 *be the inverse image of the nilpotent cone by the first projection.*

(i) *The subvariety* Γ *of* $\mathfrak{g}\times \mathrm{Gr}_\ell(\mathfrak{g})$ *has dimension* $2n+1$. *Moreover,* Γ *is contained in* Δ.

(ii) *The varieties* C_h *and* $G.X$ *are the images of* Γ *by the first and second projections respectively.*

(iii) *The subvariety* Γ_0 *of* Γ *is equidimensional of codimension* 1.

(iv) *For x nilpotent in* $\mathfrak{g}$, *the subvariety of elements V of G.X, containing x and contained in* $\overline{G(x)}$, *has dimension at most* $\dim \mathfrak{g}^x - \ell$.

Proof. (i) Since the stabilizer of $(h,\mathfrak{h})$ in $\Bbbk^*\times G$ equals $\{1\}\times H$, Γ has dimension $2n+1$. Since $tg(h)$ is in $g(\mathfrak{h})$ for all (t,g) in $\Bbbk^*\times G$ and since Δ is a closed subset of $\mathfrak{g}\times\mathrm{Gr}_\ell(\mathfrak{g})$, Γ is contained in Δ.

(ii) Since $\mathrm{Gr}_\ell(\mathfrak{g})$ is a projective variety, the image of Γ by the first projection is closed in $\mathfrak{g}$. So, it equals C_h since it is contained in C_h and since it contains the cone generated by $G.h$. Let ϖ be the canonical map from $\mathfrak{g}\setminus\{0\}$ to the projective space $\mathbb{P}(\mathfrak{g})$ and let $\widetilde{\Gamma}$ be the image of $\Gamma\cap(\mathfrak{g}\setminus\{0\})\times\mathrm{Gr}_\ell(\mathfrak{g})$ by the map $(x,V)\mapsto(\varpi(x),V)$. Since C_h is a closed cone, $\widetilde{\Gamma}$ is a closed subset of $\mathbb{P}(\mathfrak{g})\times\mathrm{Gr}_\ell(\mathfrak{g})$. Hence the image of $\tilde{\Gamma}$ by the second projection is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$. So, it equals $\overline{G.\mathfrak{h}}$ since it is contained in $\overline{G.\mathfrak{h}}$ and since it contains $G.\mathfrak{h}$. As a result, the image of Γ by the second projection equals $\overline{G.\mathfrak{h}}$ since it is contained in $\overline{G.\mathfrak{h}}$ and since it contains the image of $\widetilde{\Gamma}$ by the second projection.

(iii) The subvariety C_h of $\mathfrak{g}$ has dimension $2n+1$ and the nullvariety of p_1 in C_h is contained in $\mathfrak{N}_{\mathfrak{g}}$ since it is the nullvariety in $\mathfrak{g}$ of the polynomials $p_1,\ldots,p_\ell$. Hence $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of p_1 in C_h and Γ_0 is the nullvariety in Γ of the function $(x,V)\mapsto p_1(x)$. So Γ_0 is equidimensional of codimension 1 in Γ.

(iv) Let T be the subset of elements V of $G.X$, containing x and contained in $\overline{G(x)}$. Let denote by Γ_T the inverse image of $\overline{G.T}$ by the projection from Γ to $G.X$. Then Γ_T is contained in Γ_0. Since x is in all element of T and since Γ_T is invariant under G, the image of Γ_T by the first projection equals $\overline{G(x)}$. Hence

$$\dim\Gamma_T = \dim T + \dim\mathfrak{g} - \dim\mathfrak{g}^x$$

Since Γ_T is contained in Γ_0, Γ_T has dimension at most $\dim\mathfrak{g}-\ell$, whence the assertion. □

When $\mathfrak{g}$ is semisimple, let denote by $(G.X)_{\mathrm{u}}$ the subset of elements of $G.X$ contained in $\mathfrak{N}_{\mathfrak{g}}$.

Corollary 5.10. *Let suppose* $\mathfrak{g}$ *semisimple. Let x be in* $\mathfrak{N}_{\mathfrak{g}}$.

(i) *The variety* $(G.X)_{\mathrm{u}}$ *has dimension at most* $2n-\ell$.

(ii) *The variety* $Z_x\cap(G.X)_{\mathrm{u}}$ *has dimension at most* $\dim\mathfrak{g}^x-\ell$.

Proof. (i) Let T be an irreducible component of $(G.X)_{\mathrm{u}}$ and let Δ_T be its inverse image by the canonical projection from Δ to $G.X$. Then Δ_T is a vector bundle of rank ℓ over T. So it has dimension $\dim T+\ell$. Let Y be the projection of Δ_T onto $\mathfrak{g}$. Since T is an irreducible projective variety, Y is an irreducible closed subvariety of $\mathfrak{g}$ contained in $\mathfrak{N}_{\mathfrak{g}}$. The subvariety $(G.X)_{\mathrm{u}}$ of $G.X$ is invariant under G since it is so for $\mathfrak{N}_{\mathfrak{g}}$. Hence Δ_T and Y are G-invariant and for some y in $\mathfrak{N}_{\mathfrak{g}}$, $Y=\overline{G(y)}$. Denoting by F_y the fiber at y of the projection $\Delta_T\rightarrow Y$, V is contained in $\overline{G(y)}$ and contains y for all V in F_y. So, by Lemma 5.9,(iv),

$$\dim F_y \leq \dim\mathfrak{g}^y-\ell$$

Since the projection is G-equivariant, this inequality holds for the fibers at the elements of $G(y)$. Hence,

$$\dim\Delta_T\leq\dim\mathfrak{g}-\ell \text{ and } \dim T\leq 2n-\ell$$

(ii) Let Z be an irreducible component of $Z_x \cap (G.X)_{\mathrm{u}}$ and let T be an irreducible component of $(G.X)_{\mathrm{u}}$, containing Z. Let Δ_T and Y be as in (i). Then $G(x)$ is contained in Y and the inverse image of $\overline{G(x)}$ in Δ_T has dimension at least $\dim G(x) + \dim Z$. So, by (i),

$$\dim G(x) + \dim Z \leq \dim \mathfrak{g} - \ell$$

whence the assertion. □

Theorem 5.11. *For x in $\mathfrak{g}$, the variety of elements of $G.X$, containing x, has dimension at most* $\dim \mathfrak{g}^x - \ell$.

Proof. Let prove the theorem by induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ is commutative, $G.X = \{\mathfrak{g}\}$. If the derived Lie algebra of $\mathfrak{g}$ is simple of dimension 3, $G.X$ has dimension 2 and for x not in the centre of $\mathfrak{g}$, $\mathfrak{g}^x$ has dimension ℓ. Let suppose the theorem true for all reductive Lie algebra of dimension strictly smaller than $\dim \mathfrak{g}$. Let x be in $\mathfrak{g}$. Since $G.X$ has dimension $\dim \mathfrak{g} - \ell$, one can suppose x not in the centre of $\mathfrak{g}$. If x is not nilpotent, $\mathfrak{g}^{x_{\mathrm{s}}}$ has dimension strictly smaller than $\dim \mathfrak{g}$ and all element of $G.X$ containing x is contained in $\mathfrak{g}^{x_{\mathrm{s}}}$ by Corollary 5.3,(i), whence the theorem in this case by induction hypothesis. As a result, by Lemma 5.8, for all x in $\mathfrak{g}$, all irreducible component of Z_x, containing an element not contained in $\mathfrak{N}_{\mathfrak{g}}$, has dimension at most $\dim \mathfrak{g}^x - \ell$.

Let $\mathfrak{z}_{\mathfrak{g}}$ be the centre of $\mathfrak{g}$ and let x be a nilpotent element of $\mathfrak{g}$. Denoting by Z'_x the subset of elements of $\overline{G.(\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])}$ containing x, Z_x is the image of Z'_x by the map $V \mapsto V + \mathfrak{z}_{\mathfrak{g}}$, whence the theorem by Corollary 5.10. □

5.5. Let s be in $\mathfrak{h} \setminus \{0\}$. Let set $\mathfrak{p} := \mathfrak{g}^s + \mathfrak{b}$ and let denote by $\mathfrak{p}_{\mathrm{u}}$ the nilpotent radical of $\mathfrak{p}$. Let P be the normalizer of $\mathfrak{p}$ and let P_{u} be its unipotent radical. For a nilpotent orbit Ω of G^s in $\mathfrak{g}^s$, let denote by $\Omega^{\#}$ the induced orbit by Ω from $\mathfrak{g}^s$ to $\mathfrak{g}$.

Lemma 5.12. *Let Y be a G-invariant irreducible closed subset of $\mathfrak{g}$ and let Y' be the union of G-orbits of maximal dimension in Y. Let suppose that s is the semisimple component of an element x of Y'. Let denote by Ω the orbit of x_{n} under G^s and let set $Y_1 := \mathfrak{z} + \overline{\Omega} + \mathfrak{p}_{\mathrm{u}}$.*

(i) *The subset Y_1 of $\mathfrak{p}$ is closed and invariant under P.*

(ii) *The subset $G(Y_1)$ of $\mathfrak{g}$ is a closed subset of dimension* $\dim \mathfrak{z} + \dim G(x)$.

(iii) *For some nonempty open subset Y'' of Y', the conjugacy class of $\mathfrak{g}^{y_{\mathrm{s}}}$ under G does not depend on the element y of Y''.*

(iv) *For a good choice of x in Y'', Y is contained in $G(Y_1)$.*

Proof. (i) By [Ko63, §3.2, Lemma 5], G^s is connected and $P = P_{\mathrm{u}} G^s$. For all y in $\mathfrak{p}$ and for all g in P_{u}, $g(y)$ is in $y + \mathfrak{p}_{\mathrm{u}}$. Hence Y_1 is invariant under P since it is invariant under G^s. Moreover, it is a closed subset of $\mathfrak{p}$ since $\mathfrak{z} + \overline{\Omega}$ is a closed subset of $\mathfrak{g}^s$.

(ii) According to (i) and Lemma 1.4, $G(Y_1)$ is a closed subset of $\mathfrak{g}$. According to [CMa93, Theorem 7.1.1], $\Omega^{\#} \cap (\Omega + \mathfrak{p}_{\mathrm{u}})$ is a P-orbit and the centralizers in $\mathfrak{g}$ of its elements are contained in $\mathfrak{p}$. So, for all y in $\Omega^{\#} \cap (\Omega + \mathfrak{p}_{\mathrm{u}})$, the subset of elements g of G such that $g(y)$ is in Y_1 has dimension $\dim \mathfrak{p}$ since $g(y)$ is in $\Omega + \mathfrak{p}_{\mathrm{u}}$. As a result,

$$\dim G(Y_1) = \dim G \times_P Y_1 = \dim \mathfrak{p}_{\mathrm{u}} + \dim Y_1$$

Since $\dim \mathfrak{g}^x = \dim \mathfrak{g}^s - \dim \Omega$,

$$\begin{aligned} \dim Y_1 &= \dim \mathfrak{z} + \dim \mathfrak{p}_{\mathrm{u}} + \dim \mathfrak{g}^s - \dim \mathfrak{g}^x \\ \dim G(Y_1) &= \dim \mathfrak{z} + 2\dim \mathfrak{p}_{\mathrm{u}} + \dim \mathfrak{g}^s - \dim \mathfrak{g}^x \\ &= \dim \mathfrak{z} + \dim G(x) \end{aligned}$$

(iii) Let τ be the canonical morphism from $\mathfrak{g}$ to its categorical quotient $\mathfrak{g}//G$ under G and let Z be the closure in $\mathfrak{g}//G$ of $\tau(Y)$. Since Y is irreducible, Z is irreducible and there exists an irreducible component $\widetilde{Z}$ of the preimage of Z in $\mathfrak{h}$ whose image in $\mathfrak{g}//G$ equals Z. Since the set of conjugacy classes under G of the centralizers of the elements of $\mathfrak{h}$ in $\mathfrak{g}$ is finite, for some nonempty open subset $Z^{\#}$ of $\widetilde{Z}$, the centralizers of its elements are conjugate under G. The image of $Z^{\#}$ in $\mathfrak{g}//G$ contains a dense open subset Z' of Z. Let Y'' be the inverse image of Z' by the restriction of τ to Y'. Then Y'' is a dense open subset of Y and the centralizers in $\mathfrak{g}$ of the semisimple components of its elements are conjugate under G.

(iv) Let suppose that x is in Y''. Let Z_Y be the set of elements y of Y'' such that $\mathfrak{g}^{y_s} = \mathfrak{g}^s$. Then $G.Z_Y = Y''$. For all nilpotent orbit Ω of G^s in $\mathfrak{g}^s$, let set:

$$Y_\Omega = \mathfrak{z} + \overline{\Omega} + \mathfrak{p}_{\mathrm{u}}$$

Then Z_Y is contained in the union of the Y_Ω's. Hence Y'' is contained in the union of the $G(Y_\Omega)$'s. According to (ii), $G(Y_\Omega)$ is a closed subset of $\mathfrak{g}$. Hence Y is contained in the union of the $G(Y_\Omega)$'s since Y'' is dense in Y. Then Y is contained in $G(Y_\Omega)$ for some Ω since Y is irreducible and since there are finitely many nilpotent orbits in $\mathfrak{g}^s$, whence the assertion. □

Theorem 5.13. *(i) The variety $G.X$ is the union of $G.\mathfrak{h}$ and the $G.X_\beta$'s, $\beta \in \Pi$.*
(ii) *The variety X is the union of $U.\mathfrak{h}$ and the X_α's, $\alpha \in \mathcal{R}_+$.*

Proof. Let $\mathfrak{z}_\mathfrak{g}$ be the centre of $\mathfrak{g}$ and let μ be the map

$$\mathrm{Gr}_\ell([\mathfrak{g},\mathfrak{g}]) \longrightarrow \mathrm{Gr}_\ell(\mathfrak{g}) \qquad V \longmapsto \mathfrak{z}_\mathfrak{g} + V$$

and let set:

$$X_d := \overline{B.(\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}])} \qquad X_{\alpha,d} := \overline{B.(V_\alpha\cap[\mathfrak{g},\mathfrak{g}])}$$

for α in $\mathcal{R}_+$. Then X, $G.X$, X_α, $G.X_\alpha$ are the images of X_d, $G.X_d$, $X_{\alpha,d}$, $G.X_{\alpha,d}$ by μ respectively. So one can suppose $\mathfrak{g}$ semisimple.

(i) For $\ell = 1$, $\mathfrak{g}$ is simple of dimension 3. In this case, $G.X$ is the union of $G.\mathfrak{h}$ and $G.\mathfrak{g}^e$. So, one can suppose $\ell \geq 2$. According to Lemma 5.1,(iii), for α in $\mathcal{R}_+$, $G.X_\alpha$ is an irreducible component of $G.X\setminus G.\mathfrak{h}$. Moreover, for all β in $\Pi\cap W(\mathcal{R})(\alpha)$, $G.X_\alpha = G.X_\beta$ since V_α and V_β are conjugate under $N_G(\mathfrak{h})$.

Let T be an irreducible component of $G.X\setminus G.\mathfrak{h}$. Let set:

$$\Delta_T := \Delta\cap\mathfrak{g}\times T$$

and let denote by Y the image of Δ_T by the first projection. Then Y is closed in $\mathfrak{g}$ since $\mathrm{Gr}_\ell(\mathfrak{g})$ is a projective variety. Since Δ_T is a vector bundle over T and since T is irreducible, Δ_T is irreducible and Y is too. Since T is an irreducible component of $G.X\setminus G.\mathfrak{h}$, T, Δ_T and Y are G-invariant. According to Lemma 5.1,(iii), T has codimension 1 in $G.X$. Hence, by Corollary 5.10,(i) Y is not contained in the nilpotent cone since $\ell \geq 2$. Let Y' be the set of elements x of Y such that $\mathfrak{g}^x$ has minimal dimension. According to Lemma 5.12,(ii) and (iv), for x in a G-invariant dense subset Y'' of Y',

$$\dim Y \leq \dim G(x) + \dim \mathfrak{z}$$

with $\mathfrak{z}$ the centre of $\mathfrak{g}^{x_s}$ and according to Theorem 5.11,

$$\dim \Delta_T \leq \dim G(x) + \dim \mathfrak{z} + \dim \mathfrak{g}^x - \ell = \dim \mathfrak{g} + \dim \mathfrak{z} - \ell$$

Hence Δ_T has dimension at most $2n + \dim \mathfrak{z}$ and $\dim \mathfrak{z} = \ell - 1$ since T has codimension 1 in $G.X$. Let x be in Y'' such that x_s is in $\mathfrak{h}$. Then x_s is subregular and $\mathfrak{z}$ is the kernel of a positive root α. Denoting by $\mathfrak{s}_\alpha$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$, $\mathfrak{g}^{x_s}$ is the direct sum of $\mathfrak{h}_\alpha$ and $\mathfrak{s}_\alpha$. Since the maximal commutative subalgebras of $\mathfrak{s}_\alpha$ have dimension 1, a commutative subalgebra of dimension ℓ of $\mathfrak{g}^{x_s}$ is either a Cartan subalgebra of $\mathfrak{g}$ or conjugate to V_α under the adjoint group of $\mathfrak{g}^{x_s}$. As a result, V_α is in T and $T = \overline{G.V_\alpha} = G.X_\alpha$ since T is G-invariant, whence the assertion.

(ii) According to Lemma 5.1,(ii), for α in $\mathcal{R}_+$, X_α is an irreducible component of $X \setminus B.\mathfrak{h}$. Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ be its simple factors. For $j = 1, \ldots, m$, let denote by X_j be the closure in $\mathrm{Gr}_{\ell_{\mathfrak{g}_j}}(\mathfrak{g}_j)$ of the orbit of $\mathfrak{h} \cap \mathfrak{g}_j$. Then $X = X_1 \times \cdots \times X_m$ and the complement of $B.\mathfrak{h}$ in X is the union of the

$$X_1 \times \cdots \times X_{j-1} \times (X_j \setminus B.(\mathfrak{h} \cap \mathfrak{g}_j)) \times X_{j+1} \times \cdots \times X_m$$

So, one can suppose $\mathfrak{g}$ simple. Let consider

$$\mathfrak{b} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_\ell = \mathfrak{g}$$

an increasing sequence of parabolic subalgebras verifying the following condition: for $i = 0, \ldots, \ell - 1$, there is no parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that

$$\mathfrak{p}_i \subsetneq \mathfrak{q} \subsetneq \mathfrak{q}_{i+1}$$

For $i = 0, \ldots, \ell$, let P_i be the normalizer of $\mathfrak{p}_i$ in G, let $\mathfrak{p}_{i,\mathrm{u}}$ be the nilpotent radical of $\mathfrak{p}_i$ and let $P_{i,\mathrm{u}}$ be the unipotent radical of P_i. For $i = 0, \ldots, \ell$ and for α in $\mathcal{R}_+$, let set $X_i := P_i.X$ and $X_{i,\alpha} := P_i.X_\alpha$. Let prove by induction on $\ell - i$ that for all sequence of parabolic subalgebras verifying the above condition, the $X_{i,\alpha}$'s, $\alpha \in \mathcal{R}_+$, are the irreducible components of $X_i \setminus P_i.\mathfrak{h}$.

For $i = \ell$, it results from (i). Let suppose that it is true for $i+1$. According to Lemma 5.1,(iii), the $X_{i,\alpha}$'s are irreducible components of $X_i \setminus P_i.\mathfrak{h}$.

Claim 5.14. Let T be an irreducible component of $X_i \setminus P_i.\mathfrak{h}$ such that P_i is its stabilizer in P_{i+1}. Then $T = X_{i,\alpha}$ for some α in $\mathcal{R}_+$.

Proof. According to the induction hypothesis, T is contained in $X_{i+1,\alpha}$ for some α in $\mathcal{R}_+$. According to Lemma 5.1,(iv), T has codimension 1 in X_i so that $P_{i+1}.T$ and $X_{i+1,\alpha}$ have the same dimension. Then they are equal and T contains $\mathfrak{g}^x$ for some x in $\mathfrak{b}_{\mathrm{reg}}$ such that x_s is a subregular element belonging to $\mathfrak{h}$. Denoting by α' the positive root such that $\alpha'(x_s) = 0$, $\mathfrak{g}^x = V_{\alpha'}$ since $V_{\alpha'}$ is the commutative subalgebra contained in $\mathfrak{b}$ and containing $\mathfrak{h}_{\alpha'}$, which is not Cartan, so that $T = X_{i,\alpha'}$. □

Let suppose that $X_i \setminus P_i.\mathfrak{h}$ is not the union of the $X_{i,\alpha}$'s, $\alpha \in \mathcal{R}_+$. One expects a contradiction. Let T be an irreducible component of $X_i \setminus P_i.\mathfrak{h}$, different from $X_{i,\alpha}$ for all α. According to Claim 5.14 and according to the condition verified by the sequence, T is invariant under P_{i+1}. Moreover, according to Claim 5.14, it is so for all sequence $\mathfrak{p}'_0,\ldots,\mathfrak{p}'_\ell$ of parabolic subalgebras verifying the above condition and such that $\mathfrak{p}'_j = \mathfrak{p}_j$ for $j = 0,\ldots,i$. As a result, for all simple root β such that $\mathfrak{g}^{-\beta}$ is not in $\mathfrak{p}_i$, T is invariant under the one parameter subgroup of G generated by $\mathrm{ad}\,\mathfrak{g}^{-\beta}$. Hence T is invariant under G. It is impossible since for x in $\mathfrak{g}\setminus\{0\}$, the orbit $G(x)$ is not contained in $\mathfrak{p}_i$ since $\mathfrak{g}$ is simple, whence the assertion. □

5.6. Let X' be the subset of $\mathfrak{g}^x$ with x in $\mathfrak{b}_{\rm reg}$ such that $x_{\rm s}$ is regular or subregular. For α in $\mathcal{R}_+$, let denote by θ_α the map

$$\Bbbk \longrightarrow X \qquad t \longmapsto \exp(t\,\mathrm{ad}\,x_\alpha).\mathfrak{h}$$

According to [Sh94, Ch. VI, Theorem 1], θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$, also denoted by θ_α. Let set $Z_\alpha := \theta_\alpha(\mathbb{P}^1(\Bbbk))$ and $X'_\alpha := B.Z_\alpha$ so that $X'_\alpha = U.\mathfrak{h}\cup B.V_\alpha$.

Lemma 5.15. *Let α be in $\mathcal{R}_+$ and let V be in X. Let denote by $\overline{V}$ the image of V by the projection $x\mapsto\overline{x}$.*
(i) *For x in $\mathfrak{h}$, x is subregular if and only if $V_x = \mathfrak{h}_\gamma$ for some positive root.*
(ii) *If $\overline{V} = \mathfrak{h}_\alpha$, then $V_{\overline{x}} = \mathfrak{h}_\alpha$ for some x in V.*
(iii) *If $\overline{V} = \mathfrak{h}_\alpha$, then V is conjugate to V_α under B.*

Proof. (i) First of all, since $\varepsilon_1,\ldots,\varepsilon_\ell$ are G-equivariant maps, V_x is contained in the centre of $\mathfrak{g}^x$ for all x in $\mathfrak{g}$. Then for x in $\mathfrak{h}$, V_x is the centre of $\mathfrak{g}^x$ by Corollary 5.3,(i), whence the assertion.

(ii) Let suppose $\overline{V} = \mathfrak{h}_\alpha$. Then x is not regular semisimple for all x in V. Let suppose that $x_{\rm s}$ is not subregular for all x in V. One expects a contradiction. Since $x_{\rm s}$ and $\overline{x}$ are conjugate under B, for all x in V, there exists γ in $\mathcal{R}_+\setminus\{\alpha\}$ such that $\gamma(\overline{x}) = 0$. Hence $\overline{V}$ is contained in $\mathfrak{h}_\gamma$ for some γ in $\mathcal{R}_+\setminus\{\alpha\}$ since $\mathcal{R}_+$ is finite, whence the contradiction. Then by (i), for some x in V, $V_{\overline{x}} = \mathfrak{h}_\alpha$ since $x_{\rm s}$ and $\overline{x}$ are conjugate under B.

(iii) Let suppose $\overline{V} = \mathfrak{h}_\alpha$. By (ii), $V_{\overline{x}} = \mathfrak{h}_\alpha$ for some x in V. Let b be in B such that $b(x_{\rm s}) = \overline{x}$. Then $b(V)$ centralizes $\mathfrak{h}_\alpha$ by Corollary 5.3,(i). Moreover, $b(V)$ is not a Cartan subalgebra since $\overline{V}$ does not contain regular semsimple element. The centralizer of $\mathfrak{h}_\alpha$ in $\mathfrak{b}$ equals $\mathfrak{h}+\mathfrak{g}^\alpha$ and V_α is the commutative algebra of dimension ℓ contained in $\mathfrak{h}+\mathfrak{g}^\alpha$ which is not a Cartan subalgebra, whence the assertion. □

Corollary 5.16. *Let α be a positive root.*
(i) *The subset X'_α of X is open.*
(ii) *The subset X' of X is open. Moreover, $G.X'_\alpha$ and $G.X'$ are open subsets of $G.X$.*

Proof. (i) Since X'_α is B-invariant and since $U.\mathfrak{h}$ is an open subset of X, contained in X'_α, it suffices to prove that X'_α is a neighborhood of V_α in X. Let denote by H_α the coroot of α and let set:

$$E' := \bigoplus_{\gamma\in\mathcal{R}_+\setminus\{\alpha\}}\mathfrak{g}^\gamma \qquad E := \Bbbk H_\alpha\oplus E'$$

Let Ω_E be the set of subspaces V of $\mathfrak{b}$ such that E is a complement of V in $\mathfrak{b}$ and let Ω'_E be the complement in $X\cap\Omega_E$ of the union of the X_γ's, $\gamma\in\mathcal{R}_+\setminus\{\alpha\}$. Then Ω'_E is an open neighborhood of V_α in X. Let V be in Ω'_E such that V is not a Cartan subalgebra and let denote by $\overline{V}$ its image by the projection $x\mapsto\overline{x}$. Then V is contained in $\overline{V}+\mathfrak{u}$ so that $\mathfrak{h} = \Bbbk H_\alpha+\overline{V}$. Since V is not a Cartan subalgebra and since it is commutative, $\overline{V}\cap\mathfrak{h}_{\rm reg}$ is empty. Hence $\overline{V} = \mathfrak{h}_\gamma$ for some positive root γ. According to Lemma 5.15,(iii), V is conjugate to V_γ under B. Then $\alpha = \gamma$ and V is in X'_α since V is not in X_δ for all positive root δ different from α. As a result, Ω'_E is contained in X'_α, whence the assertion.

(ii) By definition, X' is the union the X'_α's, $\alpha \in \mathcal{R}_+$. Hence X' is an open subset of X by (i). Since X'_α is invariant under B, $X \setminus X'_\alpha$ is a B-invariant closed subset of X. Hence $G.(X \setminus X'_\alpha)$ is a closed subset of $G.X$ by Lemma 1.4. Moreover, $G.X'_\alpha$ is the complement of $G.(X \setminus X'_\alpha)$ in $G.X$. Hence $G.X'_\alpha$ and $G.X'$ are open subsets of $G.X$. □

For β in Π, let set:

$$\mathfrak{u}_\beta := \bigoplus\nolimits_{\alpha\in\mathcal{R}_+\setminus\{\beta\}} \mathfrak{g}^\alpha \qquad U_\beta := \exp(\mathrm{ad}\,\mathfrak{u}_\beta)$$

Let $\mathfrak{Y}$ be the subvariety of elements (V, V') of $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathrm{Gr}_{\ell-1}(\mathfrak{g})$ such that V' is contained in V.

Lemma 5.17. *Let β be in Π and let set: $Y_\beta := \mathfrak{Y} \cap (X'_\beta \times B.\mathfrak{h}_\beta)$.*

(i) *The variety Y_β is a smooth open subset of $\mathfrak{Y} \cap X \times B.\mathfrak{h}_\beta$.*

(ii) *The variety X'_β is smooth.*

(iii) *The subset $G.Y_\beta$ of $\mathfrak{Y}$ is the intersection of $\mathfrak{Y}$ and $G.X'_\beta \times G.\mathfrak{h}_\beta$. Moreover, the restriction to $G.Y_\beta$ of the first projection has finite fibers.*

(iv) *The canonical projection from $G.Y_\beta$ to $G.X'_\beta$ is a finite surjective morphism.*

(v) *The variety $G.Y_\beta$ is smooth.*

Proof. (i) According to Corollary 5.16, X'_β is an open subset of X. Hence Y_β is an open subset of $\mathfrak{Y} \cap X \times B.\mathfrak{h}_\beta$. By definition, $X'_\beta = B.Z_\beta$. For (g, g') in $B \times B$ and for V in Z_β, $(g(V), g'(\mathfrak{h}_\beta))$ is in $\mathfrak{Y}$ if and only if $\mathfrak{h}_\beta$ is contained in $(g')^{-1}g(V)$. Since the centralizer of $\mathfrak{h}_\beta$ in $\mathfrak{b}$ equals $\mathfrak{g}^\beta + \mathfrak{h}$ and since V is a commutative algebra, $(g')^{-1}g(V)$ is in Z_β in this case. Hence $Y_\beta = B.(Z_\beta \times \{\mathfrak{h}_\beta\})$

Let T_β be the normalizer of $\mathfrak{h}_\beta$ in B. Since $B = U_\beta T_\beta$, the map $g \mapsto g(\mathfrak{h}_\beta)$ from U_β to $B.\mathfrak{h}_\beta$ is an isomorphism. Hence the map

$$U_\beta \times Z_\beta \longrightarrow Y_\beta \qquad (g, V) \longmapsto (g(V), g(\mathfrak{h}_\beta))$$

is an isomorphism so that Y_β is smooth since Z_β is too.

(ii) Since $X'_\beta = B.Z_\beta$ and since $B.\mathfrak{h}$ is a smooth open subset of X'_β, it suffices to prove that V_β is a smooth point of X'_β. Let set:

$$F := \mathfrak{u}_\beta \oplus \Bbbk H_\beta$$

and let denote by Ω_F the set of complements of F in $\mathfrak{b}$. Then Ω_F is an affine open subset of $\mathrm{Gr}_\ell(\mathfrak{b})$, containing V_β, and the map

$$\mathrm{Hom}_\Bbbk(V_\beta, F) \longrightarrow \Omega_F \qquad \varphi \longmapsto \mathrm{span}(\{v + \varphi(v) \mid v \in V_\beta\}$$

is an isomorphism. Let denote it by χ.

Claim 5.18. Let $v_1, \ldots, v_{\ell-1}$ be a basis of $\mathfrak{h}_\beta$. Let denote by $\widetilde{\chi}$ the map

$$\mathrm{Hom}_\Bbbk(V_\beta, F) \times \Bbbk^{\ell-1} \longrightarrow \Omega_F \times \mathrm{Gr}_{\ell-1}(\mathfrak{b})$$

$$(\varphi, a_1, \ldots, a_{\ell-1}) \longmapsto (\mathrm{span}(\{v + \varphi(v) \mid v \in V_\beta\}), \mathrm{span}(\{v_i + a_i x_\beta + \varphi(v_i + a_i x_\beta) \mid i = 1, \ldots, \ell - 1\}))$$

Then $\widetilde{\chi}$ is an isomorphism onto an open neighborhood Ω'_F of $(V_\beta, \mathfrak{h}_\beta)$ in $\mathfrak{Y}$.

Proof. Let F' be the subspace of $\mathfrak{b}$ generated by F and x_β and let $\Omega_{F'}$ be the set of complements of F' in $\mathfrak{b}$. Then $\Omega_F \times \Omega_{F'}$ is an open neighborood of $(V_\beta, \mathfrak{h}_\beta)$ in $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_{\ell-1}(\mathfrak{b})$ and the map

$$\mathrm{Hom}_{\Bbbk}(V_\beta, F) \times \mathrm{Hom}_{\Bbbk}(\mathfrak{h}_\beta, F') \longrightarrow \Omega_F \times \Omega_{F'}$$

$$(\varphi, \psi) \longmapsto (\mathrm{span}(\{v + \varphi(v) \mid v \in V_\beta\}), \mathrm{span}(\{v + \psi(v) \mid v \in \mathfrak{h}_\beta\}))$$

is an isomorphism. For $(\varphi, a_1, \ldots, a_{\ell-1})$ in $\mathrm{Hom}_{\Bbbk}(V_\beta, F) \times \Bbbk^{\ell-1}$, $\widetilde{\chi}(\varphi, a_1, \ldots, a_{\ell-1})$ is the image of (φ, ψ) with ψ in $\mathrm{Hom}_{\Bbbk}(\mathfrak{h}_\beta, F')$ defined by $\psi(v_i) = \varphi(v_i + a_i x_\beta) + a_i x_\beta$ for $i = 1, \ldots, \ell - 1$. Conversely, let (φ, ψ) be such that its image is in $\mathcal{Y}$. Then for $i = 1, \ldots, \ell - 1$,

$$v_i + \psi(v_i) = \sum_{j=1}^{\ell} a_{i,j}(v_j + \varphi(v_j)) + a_{i,\ell}(x_\beta + \varphi(x_\beta))$$

with $a_{i,1}, \ldots, a_{i,\ell}$ in $\Bbbk$ so that

$$\psi(v_i) = a_{i,\ell}(x_\beta + \varphi(x_\beta)) + \varphi(v_i)$$

whence the claim since the map

$$\mathrm{Hom}_{\Bbbk}(V_\beta, F) \times \Bbbk^{\ell-1} \longrightarrow \mathrm{Hom}_{\Bbbk}(V_\beta, F) \times \mathrm{Hom}_{\Bbbk}(\mathfrak{h}_\beta, F')$$

$$(\varphi, a_1, \ldots, a_{\ell-1}) \longmapsto (\varphi, \psi) \text{ with } \psi(v_i) = \varphi(v_i + a_i x_\beta) + a_i x_\beta,\ i = 1, \ldots, \ell - 1$$

is an isomorphism onto a subspace of $\mathrm{Hom}_{\Bbbk}(V_\beta, F) \times \mathrm{Hom}_{\Bbbk}(\mathfrak{h}_\beta, F')$. □

Let identify $\mathrm{Hom}_{\Bbbk}(V_\beta, F)$ with $\mathrm{Hom}_{\Bbbk}(V_\beta, \mathfrak{u}_\beta) \times \Bbbk^\ell$ by the isomorphism

$$\mathrm{Hom}_{\Bbbk}(V_\beta, \mathfrak{u}_\beta) \times \Bbbk^\ell \longrightarrow \mathrm{Hom}_{\Bbbk}(V_\beta, F)$$

$$(\varphi, b_1, \ldots, b_\ell) \longmapsto (\sum_{j=1}^{\ell-1} t_j v_j + t_\ell x_\beta \mapsto \varphi(\sum_{j=1}^{\ell-1} t_j v_j + t_\ell x_\beta) + (\sum_{j=1}^{\ell} t_j b_j) H_\beta)$$

Let Σ be the inverse image by χ of $\Omega_F \cap X'_\beta$. Then Σ is an irreducible locally closed subset of $\mathrm{Hom}_{\Bbbk}(V_\beta, F)$ since $\Omega_F \cap X'_\beta$ is an irreducible locally closed subset of $\mathrm{Gr}_\ell(\mathfrak{b})$. Moreover,

$$\widetilde{\chi}(\Sigma \times \Bbbk^{\ell-1}) = \Omega'_F \cap \mathcal{Y} \cap X'_\beta \times \mathrm{Gr}_{\ell-1}(\mathfrak{b})$$

Let set:

$$S_0 := \{(\varphi, b_1, \ldots, b_\ell, a_1, \ldots, a_{\ell-1}) \mid (\varphi, b_1, \ldots, b_\ell) \in \Sigma,\ b_i + b_\ell a_i = 0,\ i = 1, \ldots, \ell - 1\}$$

Claim 5.19. Let S be the inverse image of $\Omega'_F \cap Y_\beta$ by $\widetilde{\chi}$. Then S is an irreducible subvariety of S_0. Moreover, Σ is the image of S by the canonical projection from $\mathrm{Hom}_{\Bbbk}(V_\beta, F) \times \Bbbk^{\ell-1}$ to $\mathrm{Hom}_{\Bbbk}(V_\beta, \Bbbk^{\ell-1})$.

Proof. Since $Y_\beta = B.(Z_\beta \times \{\mathfrak{h}_\beta\})$, Y_β and $\Omega'_F \cap Y_\beta$ are irreducible varieties. Hence by Claim 5.18, S is an irreducible variety. Moreover, $\widetilde{\chi}(S) = \Omega'_F \cap Y_\beta$ and Σ is the image of S by the projection onto $\mathrm{Hom}_{\Bbbk}(V_\beta, F)$ since X'_β is the image of Y_β by the projection from $\mathcal{Y}$ to $\mathrm{Gr}_\ell(\mathfrak{b})$ and since $\Sigma = \chi^{-1}(\Omega_F \cap X'_\beta)$. Let $(\varphi, b_1, \ldots, b_\ell, a_1, \ldots, a_{\ell-1})$ be in S. Then $\chi(\varphi, b_1, \ldots, b_\ell)$ is in $\Omega_F \cap X'_\beta$ and for $i = 1, \ldots, \ell - 1$,

$$\overline{\varphi(v_i + a_i x_\beta) + (b_i + b_\ell a_i) H_\beta + a_i x_\beta} \in \mathfrak{h}_\beta$$

Hence S is contained in S_0. □

Let suppose that S is not contained in $\Sigma \times \{0\}$. One expects a contradiction. Since $\Omega_F \cap X'_\beta$ contains Cartan subalgebras and since Σ is irreducible, for all $(\varphi, b_1, \dots, b_\ell)$ in a dense subset Σ' of Σ, $(b_1, \dots, b_\ell) \neq 0$. Then, by Claim 5.19, $b_\ell \neq 0$. Let set $S' := S \cap \Sigma' \times \Bbbk^{\ell-1}$ and let $(\varphi, b_1, \dots, b_\ell, a_1, \dots, a_{\ell-1})$ be in S' such that $(a_1, \dots, a_{\ell-1}) \neq 0$. After a permutation of the v_i's, one can suppose $a_1 \neq 0$ so that $b_1 \neq 0$. Then

$$0 = [v_1 + \varphi(v_1) + b_1 H_\beta, x_\beta + \varphi(x_\beta) + b_\ell H_\beta] \in 2b_1 x_\beta + \mathfrak{u}_\beta$$

whence the contradiction. As a result, $S = \Sigma \times \{0\}$. By (i), S is a smooth variety. Hence Σ is a smooth variety and $\Omega_F \cap X'_\beta$ is a smooth open subset of X'_β, containing V_β, whence the assertion.

(iii) Since $\mathfrak{Y}$ is G-invariant, $G.Y_\beta$ is contained in $\mathfrak{Y} \cap G.X'_\beta \times G.\mathfrak{h}_\beta$. Let (V, V') be in this intersection. If V is not a Cartan subalgebra, for some g in G, $g(V) = V_\beta$ and $g(V') = \mathfrak{h}_\beta$ since $\mathfrak{h}_\beta$ is the set of semisimple elements contained in V_β. Let suppose that V is a Cartan subalgebra, for some g in G, $g(V) = \mathfrak{h}$ and $g(V')$ is an element of $G.\mathfrak{h}_\beta$ contained in $\mathfrak{h}$. In particular, $g(V')$ contains a subregular element. So, $g(V') = \mathfrak{h}_\alpha$ for some positive root α. Moreover, $w(\alpha) = \beta$ for some w in $W(\mathcal{R})$ since $g(V')$ is in $G.\mathfrak{h}_\beta$, whence $wg(V') = \mathfrak{h}_\beta$ and (V, V') is in $G.Y_\beta$, whence the assertion.

(iv) According to (iii), it suffices to prove

$$\overline{G.Y_\beta} \cap G.X'_\beta \times \mathrm{Gr}_{\ell-1}(\mathfrak{g}) \subset G.X'_\beta \times G.\mathfrak{h}_\beta$$

since $\overline{G.Y_\beta}$ is a projective variety. According to (i) and Lemma 1.4, $\overline{G.Y_\beta} = G.\overline{B.(Z_\beta \times \{\mathfrak{h}_\beta\})}$. Let (V, V') be in $\overline{G.Y_\beta}$ such that V is in X'_β. Then, for some g in G, $(g(V), g(V'))$ is in $\overline{B.(Z_\beta \times \{\mathfrak{h}_\beta\})}$ so that $g(V')$ is contained in $\mathfrak{h}_\beta + \mathfrak{u}_\beta$. According to (i), for some b in B, $bg(V)$ is in Z_β and $bg(V')$ is contained in $(\mathfrak{h}_\beta + \mathfrak{u}_\beta) \cap (\mathfrak{h} + \mathfrak{g}^\beta)$. Hence $bg(V') = \mathfrak{h}_\beta$ and V' is in $G.\mathfrak{h}_\beta$, whence the assertion.

(v) Let denote by $\mathfrak{s}_\beta$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^\beta$ and $\mathfrak{g}^{-\beta}$. Let T'_β be the normalizer of $\mathfrak{h}_\beta$ in G and let Z'_β be the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under T'_β. Since the normalizer of $\mathfrak{h}_\beta$ in $\mathfrak{g}$ equals $\mathfrak{h} + \mathfrak{s}_\beta$, Z'_β is the set of subspaces of $\mathfrak{g}$ generated by $\mathfrak{h}_\beta$ and an element of $\mathfrak{s}_\beta \setminus \{0\}$ so that Z'_β is isomorphic to $\mathbb{P}^2(\Bbbk)$. Moreover, $G.Y_\beta$ equals $G.(Z'_\beta \times \{\mathfrak{h}_\beta\})$ since $G.Y_\beta = G.(Z_\beta \times \{\mathfrak{h}_\beta\})$ by (i), and one has a commutative diagram

$$\begin{tikzcd}
G \times_{T'_\beta} (Z'_\beta \times \{\mathfrak{h}_\beta\}) \arrow[rr, hook] \arrow[dr] & & G/T'_\beta \times G.Y_\beta \arrow[dl] \\
& G.Y_\beta \arrow[ul, dotted, "\phi"'] &
\end{tikzcd}$$

The canonical projection $G.Y_\beta \to G.\mathfrak{h}_\beta$ gives a morphism $G.Y_\beta \to G/T'_\beta$ whence an inverse φ of the diagonal arrow. Hence $G.Y_\beta$ is isomorphic to $G \times_{T'_\beta} (Z'_\beta \times \{\mathfrak{h}_\beta\})$ so that $G.Y_\beta$ is smooth since G/T'_β and $Z'_\beta \times \{\mathfrak{h}_\beta\}$ are smooth, whence the assertion. □

Let denote by X_{n} and $(G.X)_{\mathrm{n}}$ the normalizations X of $G.X$ and let denote by θ_0 and θ the normalization morphisms $X_{\mathrm{n}} \to X$ and $(G.X)_{\mathrm{n}} \to G.X$ respectively.

Proposition 5.20. (i) *The open subset* $\theta^{-1}(G.X')$ *of* $(G.X)_{\mathrm{n}}$ *is smooth and the restriction of* θ *to* $\theta^{-1}(G.X')$ *is a homeomorphism onto* $G.X'$.

(ii) *The open subset* $\theta_0^{-1}(X')$ *of* X_{n} *is smooth and the restriction of* θ_0 *to* $\theta_0^{-1}(X')$ *is a homeomorphism onto* X'.

Proof. (i) By definition, X' is the union of the X'_α's, $\alpha \in \mathcal{R}_+$. Then, since all orbit of $W(\mathcal{R})$ in $\mathcal{R}$ has a nonempty intersection with Π, $G.X'$ is the union of the $G.X'_\beta$'s, $\beta \in \Pi$. So, it suffices to prove that for β in Π, $\theta^{-1}(G.X'_\beta)$ is smooth and the restriction of θ to $\theta^{-1}(G.X'_\beta)$ is injective since θ is closed and surjective as a finite dominant morphism.

Since $G.\mathfrak{h}$ is a smooth open subset of $G.X$, the restriction of θ to $\theta^{-1}(G.\mathfrak{h})$ is an isomorphism onto $G.\mathfrak{h}$. The variety $G.V_\beta$ is an hypersurface of $G.X'_\beta$. Hence $\theta^{-1}(G.V_\beta)$ is an hypersurface of the normal variety $\theta^{-1}(G.X'_\beta)$ and its elements are smooth points of $\theta^{-1}(G.X'_\beta)$ since $\theta^{-1}(G.X'_\beta)$ is a G-variety and since a normal variety is smooth in codimension 1. As a result, $\theta^{-1}(G.X'_\beta)$ is smooth since $G.X'_\beta$ is the union of $G.\mathfrak{h}$ and $G.V_\beta$. Let x_1 and x_2 be in $\theta^{-1}(X'_\beta)$ such that $\theta(x_1) = \theta(x_2) = V_\beta$. According to Lemma 5.17,(iv) and (v), the canonical projection from $G.Y_\beta$ to $G.X'_\beta$ factorizes through the restriction of θ to $\theta^{-1}(G.X'_\beta)$ since $\theta^{-1}(G.X'_\beta)$ is the normalization of $G.X'_\beta$, whence a commutative digram

$$\begin{tikzcd} G.Y_\beta \arrow[rr, "\theta_{\mathrm{n}}"] \arrow[dr] & & \theta^{-1}(G.X'_\beta) \arrow[dl, "\theta"'] \\ & G.X'_\beta & \end{tikzcd}$$

with θ_{n} finite and surjective. Let y_1 and y_2 be in $G.Y'_\beta$ such that $\theta_{\mathrm{n}}(y_j) = x_j$ for $j = 1, 2$. Since V_β is the image of y_1 and y_2 by the canonical projection onto $G.X'_\beta$ and since $\mathfrak{h}_\beta$ is the set of semisimple elements contained in V_β, $y_1 = y_2$ and $x_1 = x_2$. Hence the restriction of θ to $\theta^{-1}(G.X'_\beta)$ is injective since θ is G-equivariant and since $G.X'_\beta$ is the union of $G.\mathfrak{h}$ and $G.V_\beta$.

(ii) According to Corollary 5.16,(ii), $\theta_0^{-1}(X')$ is an open subset of X_{n}. Since X' is the union of the X'_α's, $\alpha \in \mathcal{R}_+$, it suffices to prove that for α in $\mathcal{R}_+$, $\theta_0^{-1}(X'_\alpha)$ is smooth and the restriction of θ_0 to $\theta_0^{-1}(X'_\alpha)$ is injective since θ_0 is closed and surjective as a finite dominant morphism.

Let α be in $\mathcal{R}_+$ and let β in Π such that β is in the orbit of α under $W(\mathcal{R})$. Since $B.\mathfrak{h}$ is a smooth open subset of $B.X$, the restriction of θ_0 to $\theta_0^{-1}(B.\mathfrak{h})$ is an isomorphism onto $B.\mathfrak{h}$. The variety $B.V_\alpha$ is an hypersurface of X'_α. Hence $\theta_0^{-1}(B.V_\alpha)$ is an hypersurface of the normal variety $\theta_0^{-1}(X'_\alpha)$ and its elements are smooth points of $\theta_0^{-1}(X'_\alpha)$ since $\theta_0^{-1}(X'_\alpha)$ is a B-variety and since a normal variety is smooth in codimension 1. As a result, $\theta_0^{-1}(X'_\alpha)$ is smooth since X'_α is the union of $B.\mathfrak{h}$ and $B.V_\alpha$. Since β is in the orbit of α under $W(\mathcal{R})$, $G.X'_\alpha = G.X'_\beta$. Moreover, the varieties $G \times_B \theta_0^{-1}(X'_\beta)$ and $G \times_B \theta_0^{-1}(X'_\alpha)$ are smooth as fiber bundles over a smooth variety with smooth fibers, whence a commutative diagram

$$\begin{tikzcd} G \times_B \theta_0^{-1}(X'_\beta) \arrow[r] \arrow[d] & \theta^{-1}(G.X'_\beta) \arrow[d, "\theta"'] & G \times_B \theta_0^{-1}(X'_\alpha) \arrow[l] \arrow[d] \\ G \times_B X'_\beta \arrow[r] & G.X'_\beta & G \times_B X'_\alpha \arrow[l] \end{tikzcd}$$

by [H77, Ch. II, Proposition 4.1]. By Lemma 1.4, the horizontal arrows are projective morphisms. Indeed, since a regular element is contained in finitely many Borel subalgebras, their fibers are finite so that they are finite. Since $B.\mathfrak{h}$ is an open subset of X'_α and X'_β, $G \times_B \theta_0^{-1}(X'_\beta)$ and $G \times_B \theta_0^{-1}(X'_\alpha)$ have the same field of rational functions. As a result, since these two varieties are normal, there exists a G-equivariant isomorphism from $G\times_B \theta_0^{-1}(X'_\beta)$ onto $G\times_B \theta_0^{-1}(X'_\alpha)$ by [H77, Ch. II, Proposition 4.1]. According to Lemma 5.17,(ii), the restriction of θ_0 to $\theta_0^{-1}(X'_\beta)$ is an isomorphism so that the first down arrow in the above diagram is an isomorphism. Moreover, the restriction to all fiber of $G\times_B \theta_0^{-1}(X'_\beta)$ of the morphism

$$G\times_B \theta_0^{-1}(X'_\beta) \longrightarrow \theta^{-1}(G.X'_\beta)$$

is injective. Hence the restriction of θ_0 to $\theta_0^{-1}(X'_\alpha)$ is injective since the restriction of θ to $\theta^{-1}(G.X'_\beta)$ is too by (ii), whence the assertion. □

6. On the generalized isospectral commuting variety.

Let $k\geq 2$ be an integer. Let denote by $\mathcal{C}^{(k)}$ the closure of $G.\mathfrak{h}^k$ in $\mathfrak{g}^k$ with respect to the diagonal action of G in $\mathfrak{g}^k$ and let set $\mathcal{C}_{\mathrm{n}}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$. The varieties $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\mathrm{n}}^{(k)}$ are called generalized commuting variety and generalized isospectral commuting variety respectively. For $k=2$, $\mathcal{C}_{\mathrm{n}}^{(k)}$ is the isospectral commuting variety considered by M. Haiman in [Ha99, §8] and [Ha02, §7.2].

6.1. Let set:

$$E^{(k)} := \{(u,x_1,\dots,x_k)\in X\times\mathfrak{b}^k \mid u\ni x_1,\dots,u\ni x_k\}$$

Lemma 6.1. *Let denote by $E^{(k,*)}$ the intersection of $E^{(k)}$ and $U.\mathfrak{h}\times(\mathfrak{g}_{\mathrm{reg,ss}}\cap\mathfrak{b})^k$ and for w in $W(\mathcal{R})$, let denote by θ_w the map*

$$E^{(k)} \longrightarrow \mathfrak{b}^k\times\mathfrak{h}^k \qquad (u,x_1,\dots,x_k)\longmapsto (x_1,\dots,x_k,w(\overline{x_1}),\dots,w(\overline{x_k}))$$

(i) *Denoting by $\mathfrak{X}_{0,k}$ the image of $E^{(k)}$ by the projection $(u,x_1,\dots,x_k)\mapsto(x_1,\dots,x_k)$, $\mathfrak{X}_{0,k}$ is the closure of $B.\mathfrak{h}^k$ in $\mathfrak{b}^k$ and $\mathcal{C}^{(k)}$ is the image of $G\times\mathfrak{X}_{0,k}$ by the map $(g,x_1,\dots,x_k)\mapsto(g(x_1),\dots,g(x_k))$.*

(ii) *For all w in $W(\mathcal{R})$, $\theta_w(E^{(k,*)})$ is dense in $\theta_w(E^{(k)})$.*

Proof. (i) Since X is a projective variety, $\mathfrak{X}_{0,k}$ is a closed subset of $\mathfrak{b}^k$. The variety $E^{(k)}$ is irreducible of dimension $n+\Bbbk\ell$ as a vector bundle of rank $\Bbbk\ell$ over the irreducible variety X. So, $B.(\{\mathfrak{h}\}\times\mathfrak{h}^k)$ is dense in $E^{(k)}$ and $\mathfrak{X}_{0,k}$ is the closure of $B.\mathfrak{h}^k$ in $\mathfrak{b}^k$, whence the assertion by Lemma 1.4.

(ii) Since $U.\mathfrak{h}\times(\mathfrak{g}_{\mathrm{reg,ss}}\cap\mathfrak{b})^k$ is an open susbet of $X\times\mathfrak{b}^k$, $E^{(k,*)}$ is an open subset of $E^{(k)}$. Moreover, it is a dense open subset since $E^{(k)}$ is irreducible as a vector bundle over the irreducible variety X, whence the assertion since θ_w is a morphism of algebraic varieties. □

6.2. Let s be in $\mathfrak{h}$ and let G^s be the centralizer of s in G. According to [Ko63, §3.2, Lemma 5], G^s is connected. Let denote by $\mathcal{R}_s$ the set of roots whose kernel contains s and let denote by $W(\mathcal{R}_s)$ the Weyl group of $\mathcal{R}_s$. Let $\mathfrak{z}_s$ be the centre of $\mathfrak{g}^s$.

Lemma 6.2. *Let $x=(x_1,\dots,x_k)$ be in $\mathcal{C}^{(k)}$ verifying the following conditions:*

(1) *s is the semisimple component of x_1,*

(2) *for z in P_x, the centralizer in $\mathfrak{g}$ of the semisimple component of z has dimension at least $\dim\mathfrak{g}^s$.*

Then for $i=1,\dots,k$, the semisimple component of x_i is contained in $\mathfrak{z}_s$.

Proof. Since x is in $\mathcal{C}^{(k)}$, $[x_i, x_j] = 0$ for all (i, j). Let suppose that for some i, the semisimple component $x_{i,\mathrm{s}}$ of x_i is not in $\mathfrak{z}_s$. One expects a contradiction. Since $[x_1, x_i] = 0$, for all t in $\Bbbk$, $s+tx_{i,\mathrm{s}}$ is the semisimple component of $x_1 + tx_i$. Moreover, after conjugation by an element of G^s, one can suppose that $x_{i,\mathrm{s}}$ is in $\mathfrak{h}$. Since $\mathcal{R}$ is finite, there exists t in $\Bbbk^*$ such that the subset of roots whose kernel contains $s + tx_{i,\mathrm{s}}$ is contained in $\mathcal{R}_s$. Since $x_{i,\mathrm{s}}$ is not in $\mathfrak{z}_s$, for some α in $\mathcal{R}_s$, $\alpha(s + tx_{i,\mathrm{s}}) \neq 0$ that is $\mathfrak{g}^{s+tx_{i,\mathrm{s}}}$ is strictly contained in $\mathfrak{g}^s$, whence the contradiction. □

For w in $W(\mathcal{R})$, let set:

$$C_w := G^s wB/B \qquad B^w := wBw^{-1}$$

The following lemma results from [Hu95, §6.17, Lemma].

Lemma 6.3. *Let $\mathfrak{B}$ be the set of Borel subalgebras of $\mathfrak{g}$ and let $\mathfrak{B}_s$ be the set of Borel subalgebras of $\mathfrak{g}$ containing s.*

(i) *For all w in $W(\mathcal{R})$, C_w is a connected component of $\mathfrak{B}_s$.*

(ii) *For (w, w') in $W(\mathcal{R}) \times W(\mathcal{R})$, $C_w = C_{w'}$ if and only if $w'w^{-1}$ is in $W(\mathcal{R}_s)$.*

(iii) *The variety C_w is isomorphic to $G^s/(G^s \cap B^w)$.*

For x in $\mathcal{B}^{(k)}$, let denote by $\mathfrak{B}_x$ the subset of Borel subalgebras containing P_x.

Corollary 6.4. *Let $x = (x_1, \ldots, x_k)$ be in $\mathcal{C}^{(k)}$. Let suppose that x verifies Conditions* (1) *and* (2) *of Lemma* 6.2*. Then the $C_w \cap \mathfrak{B}_x$'s, w in $W(\mathcal{R})$ are the connected components of $\mathfrak{B}_x$.*

Proof. Since a Borel subalgebra contains the semisimple component of its elements and since s is the semisimple component of x_1, $\mathfrak{B}_x$ is contained in $\mathfrak{B}_s$. As a result, according to Lemma 6.3,(i), every connected component of $\mathfrak{B}_x$ is contained in C_w for some w in $W(\mathcal{R})$. Let set $x_{\mathrm{n}} := (x_{1,\mathrm{n}}, \ldots, x_{k,\mathrm{n}})$. Since $[x_i, x_j] = 0$ for all (i, j), P_x is contained in $\mathfrak{g}^s$. Let $\mathfrak{B}^s$ be the set of Borel subalgebras of $\mathfrak{g}^s$ and for y in $(\mathfrak{g}^s)^k$, let $\mathfrak{B}^s_y$ be the set of Borel subalgebras of $\mathfrak{g}^s$ containing P_y. According to [Hu95, Theorem 6.5], $\mathfrak{B}^s_{x_{\mathrm{n}}}$ is connected. Moreover, according to Lemma 6.2, the semisimple components of $x_1, \ldots, x_k$ are contained in $\mathfrak{z}_s$ so that $\mathfrak{B}^s_{x_{\mathrm{n}}} = \mathfrak{B}^s_x$. Let w be in $W(\mathcal{R})$. According to Lemma 6.3,(iii), there is an isomorphism from $\mathfrak{B}^s$ to C_w. Moreover, the image of $\mathfrak{B}^s_x$ by this isomorphism equals $C_w \cap \mathfrak{B}_x$, whence the corollary. □

Corollary 6.5. *Let $x = (x_1, \ldots, x_k)$ be in $\mathcal{C}^{(k)}$ verifying Conditions* (1) *and* (2) *of Lemma* 6.2*. Then $\eta^{-1}(x)$ is contained in the set of the $(x_1, \ldots, x_k, w(x_{1,\mathrm{s}}), \ldots, w(x_{k,\mathrm{s}}))$'s with w in $W(\mathcal{R})$.*

Proof. Since $\gamma = \eta \circ \gamma_{\mathrm{n}}$, $\eta^{-1}(x)$ is the image of $\gamma^{-1}(x)$ by γ_{n}. Furthermore, γ_{n} is constant on the connected components of $\gamma^{-1}(x)$ since $\eta^{-1}(x)$ is finite. Let C be a connected component of $\gamma^{-1}(x)$. Identifying $G \times_B \mathfrak{b}^k$ with the subvariety of elements (u, x) of $\mathfrak{B} \times \mathfrak{g}^k$ such that P_x is contained in u, C identifies with $C_w \cap \mathfrak{B}_x \times \{x\}$ for some w in $W(\mathcal{R})$ by Corollary 6.4. Then for some g in G^s and for some representative g_w of w in $N_G(\mathfrak{h})$, $gg_w(\mathfrak{b})$ contains P_x so that

$$\gamma_{\mathrm{n}}(C) = \{(x_1, \ldots, x_k, \overline{(gg_w)^{-1}(x_1)}, \ldots, \overline{(gg_w)^{-1}(x_k)})\}$$

By Lemma 6.2, $x_{1,\mathrm{s}}, \ldots, x_{k,\mathrm{s}}$ are in $\mathfrak{z}_s$ so that $w^{-1}(x_{i,\mathrm{s}})$ is the semisimple component of $(gg_w)^{-1}(x_i)$ for $i = 1, \ldots, k$. Hence

$$\gamma_{\mathrm{n}}(C) = \{(x_1, \ldots, x_k, w^{-1}(x_{1,\mathrm{s}}), \ldots, w^{-1}(x_{k,\mathrm{s}}))\}$$

whence the corollary. □

Proposition 6.6. *The variety* $\mathcal{C}_{\rm n}^{(k)}$ *is irreducible and equal to the closure of* $G.\iota_{\rm n}(\mathfrak{h}^k)$ *in* $\mathcal{B}_{\rm n}^{(k)}$.

Proof. Let denote by $\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$ the closure of $G.\iota_{\rm n}(\mathfrak{h}^k)$ in $\mathcal{B}_{\rm n}^{(k)}$. Since η is G-equivariant, $\eta(G.\iota_{\rm n}(\mathfrak{h}^k)) = G.\mathfrak{h}^k$. Hence $\eta(\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}) = \mathcal{C}^{(k)}$ since η is a finite morphism and since $\mathcal{C}^{(k)}$ is the closure of $G.\mathfrak{h}^k$ in $\mathfrak{g}^k$ by definition. Moreover, $\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$ is irreducible as the closure of an irreducible set. So, it suffices to prove $\mathcal{C}_{\rm n}^{(k)} = \overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$. In other words, for all x in $\mathcal{C}^{(k)}$, $\eta^{-1}(x)$ is contained in $\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$. According to Lemma 3.9,(ii), $\mathcal{B}_{\rm n}^{(k)}$ is a ${\rm GL}_k(\Bbbk)$-variety and η is ${\rm GL}_k(\Bbbk)$-equivariant. As a result, since $\mathcal{C}^{(k)}$ is invariant under ${\rm GL}_k(\Bbbk)$, for x in $\mathcal{C}^{(k)}$, $\eta^{-1}(x')$ is contained in $\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$ for all x' in P_x^k such that $P_{x'} = P_x$ if $\eta^{-1}(x)$ is contained in $\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$. Then, according to Lemma 6.2, since η is G-equivariant, it suffices to prove that $\eta^{-1}(x)$ is contained in $\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$ for x in $\mathcal{C}^{(k)} \cap \mathfrak{b}^k$ verifying Conditions (1) and (2) of Lemma 6.2 for some s in $\mathfrak{h}$.

According to Corollary 6.5,

$$\eta^{-1}(x) \subset \{(x_1,\dots,x_k,w(x_{1,{\rm s}}),\dots,w(x_{k,{\rm s}})) \mid w \in W(\mathcal{R})\} \text{ with } x = (x_1,\dots,x_k)$$

For s regular, P_x is contained in $\mathfrak{h}$ and $x_i = x_{i,{\rm s}}$ for $i = 1,\dots,k$. By definition,

$$(w(x_1),\dots,w(x_k),w(x_1),\dots,w(x_k)) \in \iota_{\rm n}(\mathfrak{h}^k)$$

and for g_w a representative of w in $N_G(\mathfrak{h})$,

$$g_w^{-1}.(w(x_1),\dots,w(x_k),w(x_1),\dots,w(x_k)) = (x_1,\dots,x_k,w(x_1),\dots,w(x_k))$$

Hence $\eta^{-1}(x)$ is contained in $G.\iota_{\rm n}(\mathfrak{h}^k)$. As a result, according to the notations of Lemma 6.1, for all w in $W(\mathcal{R})$, $\theta_w(E^{(k,*)})$ is contained in $G.\iota_{\rm n}(\mathfrak{h}^k)$. Hence, by Lemma 6.1,(ii), $\theta_w(E^{(k)})$ is contained in $\overline{G.\iota_{\rm n}(\mathfrak{h}^k)}$, whence the proposition. □

6.3. According to Corollary 3.8,(iii), the variety $\varpi^{-1}(\mathcal{B}^{(k)})$ is invariant under the action of $W(\mathcal{R})^k$ in $\mathcal{X}^k$ and according to Proposition 3.10, $\mathcal{B}_{\rm n}^{(k)}$ is an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$ and η is the restriction of ϖ to $\mathcal{B}_{\rm n}^{(k)}$.

Lemma 6.7. *Let* Φ *be the restriction to* ${\rm S}(\mathfrak{h})^{\otimes k}$ *of the canonical map from* $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$ *to* $\Bbbk[\mathcal{C}_{\rm n}^{(k)}]$.

(i) *The subvariety* $\mathcal{C}_{\rm n}^{(k)}$ *of* $\mathcal{X}^k$ *is invariant under the diagonal action of* $W(\mathcal{R})$ *in* $\mathcal{X}^k$.

(ii) *The map* Φ *is an embedding of* ${\rm S}(\mathfrak{h})^{\otimes k}$ *into* $\Bbbk[\mathcal{C}_{\rm n}^{(k)}]$. *Moreover,* $\Phi({\rm S}(\mathfrak{h})^{\otimes k})$ *equals* $\Bbbk[\mathcal{C}_{\rm n}^{(k)}]^G$.

(iii) *The image of* $({\rm S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ *by* Φ *equals* $\Bbbk[\mathcal{C}^{(k)}]^G$.

Proof. (i) For all w in $W(\mathcal{R})$ and for all representative g_w of w in $W(\mathcal{R})$,

$$(x_1,\dots,x_k,w(x_1),\dots,w(x_k)) = g_w^{-1}.(w(x_1),\dots,w(x_k),w(x_1),\dots,w(x_k))$$

for all $(x_1,\dots,x_k)$ in $\mathfrak{h}^k$. As a result, for all w in $W(\mathcal{R})$, $w.\iota_{\rm n}(\mathfrak{h}^k)$ is contained in $G.\iota_{\rm n}(\mathfrak{h}^k)$. Hence $G.\iota_{\rm n}(\mathfrak{h}^k)$ is invariant under the diagonal action of $W(\mathcal{R})$ in $\mathcal{X}^k$ since the actions of G and $W(\mathcal{R})^k$ in $\mathcal{X}^k$ commute, whence the assertion.

(ii) According to Corollary 3.12,(i), ${\rm S}(\mathfrak{h})^{\otimes k}$ equals $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]^G$. Moreover, for all P in ${\rm S}(\mathfrak{h})^{\otimes k}$ and for all x in $\mathfrak{h}^k$, $P\circ\iota_{\rm n}(x) = P(x)$. Hence Φ is injective by Proposition 6.6. Since G is reductive, $\Bbbk[\mathcal{C}_{\rm n}^{(k)}]^G$ is the image of $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]^G$ by the quotient morphism, whence the assertion.

(iii) Since G is reductive, $\Bbbk[\mathcal{C}^{(k)}]^G$ is the image of $\Bbbk[\mathcal{B}^{(k)}]^G$ by the quotient morphism, whence the assertion since $({\rm S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ equals $\Bbbk[\mathcal{B}^{(k)}]^G$ by Corollary 3.12,(ii). □

Let identify $\mathrm{S}(\mathfrak{h})^{\otimes k}$ to a subalgebra of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ by Φ.

Proposition 6.8. *Let $\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}$ and $\widetilde{\mathcal{C}^{(k)}}$ be the normalizations of $\mathcal{C}_{\mathrm{n}}^{(k)}$ and $\mathcal{C}^{(k)}$.*

(i) *The variety $\mathcal{C}^{(k)}$ is the categorical quotient of $\mathcal{C}_{\mathrm{n}}^{k}$ under the action of $W(\mathcal{R})$.*

(ii) *The variety $\widetilde{\mathcal{C}^{(k)}}$ is the categorical quotient of $\widetilde{\mathcal{C}_{\mathrm{n}}^{k}}$ under the action of $W(\mathcal{R})$.*

Proof. (i) According to Proposition 3.10,(iii), $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Since $\mathcal{C}_{\mathrm{n}}^{(k)} = \eta^{-1}(\mathcal{C}^{(k)})$ by Proposition 6.6, the image of $\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ by the restriction morphism equals $\Bbbk[\mathcal{C}^{(k)}]$. Hence $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ is generated by $\Bbbk[\mathcal{C}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Then, by Lemma 6.7,(iii), $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]^{W(\mathcal{R})} = \Bbbk[\mathcal{C}^{(k)}]$.

(ii) Let K be the fraction field of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$. Since $\mathcal{C}_{\mathrm{n}}^{(k)}$ is a $W(\mathcal{R})$-variety, there is an action of $W(\mathcal{R})$ in K and $K^{W(\mathcal{R})}$ is the fraction field of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]^{W(\mathcal{R})}$ since $W(\mathcal{R})$ is finite. As a result, the integral closure $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}]$ of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ in K is invariant under $W(\mathcal{R})$ and $\Bbbk[\widetilde{\mathcal{C}^{(k)}}]$ is contained in $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}]$. Let a be in $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}]^{W(\mathcal{R})}$. Then a verifies a dependence integral equation over $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$,

$$a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0$$

whence

$$a^m + (\frac{1}{|W(\mathcal{R})|}\sum_{w\in W(\mathcal{R})} w.a_{m-1})a^{m-1} + \cdots + \frac{1}{|W(\mathcal{R})|}\sum_{w\in W(\mathcal{R})} w.a_0 = 0$$

since a in invariant under $W(\mathcal{R})$ so that a is in $\Bbbk[\widetilde{\mathcal{C}^{(k)}}]$, whence the assertion. □

7. Desingularization.

Let $k \geq 2$ be an integer. Let X, X', X_{n}, θ_0 be as in Subsection 5.6. Let denote by X'_{n} the inverse image of X' in X_{n}. According to Proposition 5.20, X'_{n} is a smooth open subset of X_{n} and according to [Hir64], there exists a desingularization $(\Gamma, \pi_{\mathrm{n}})$ of X_{n} such that the restriction of π_{n} to $\pi_{\mathrm{n}}^{-1}(X'_{\mathrm{n}})$ is an isomorphism onto X'_{n}. Let set $\pi = \theta_0\circ\pi_{\mathrm{n}}$ so that (Γ,π) is a desingularization of X. Let recall that $\mathfrak{X}_{0,k}$ is the closure in $\mathfrak{b}^k$ of $B.\mathfrak{h}^k$ and let set $\mathfrak{X}_k := G\times_B \mathfrak{X}_{0,k}$. Then $\mathfrak{X}_k$ is a closed subvariety of $G\times_B \mathfrak{b}^k$.

Lemma 7.1. *Let E be the restriction to X of the tautological vector bundle of rank ℓ over $\mathrm{Gr}_\ell(\mathfrak{b})$ and let τ' be the canonical morphism from E to $\mathfrak{b}$.*

(i) *The morphism τ' is projective and birational.*

(ii) *Let ν be the canonical map from $\pi^*(E)$ to E. Then $\tau := \tau'\circ\nu$ is a B-equivariant birational projective morphism from $\pi^*(E)$ to $\mathfrak{b}$. In particular, $\pi^*(E)$ is a desingularization of $\mathfrak{b}$.*

Proof. (i) By definition, E is the subvariety of elements (u,x) of $X\times\mathfrak{b}$ such that x is in u so that τ' is the projection from E to $\mathfrak{b}$. Since X is a projective variety, τ' is a projective morphism and $\tau'(E)$ is closed in $\mathfrak{b}$. Moreover, $\tau'(E)$ is B-invariant since τ' is a B-equivariant morphism and it contains $\mathfrak{h}$ since $\mathfrak{h}$ is in X. As a result, $\tau'(E) = \mathfrak{b}$. By (i), for x in $\mathfrak{h}_{\mathrm{reg}}$, $(\tau')^{-1}(x) = \{(\mathfrak{h},x)\}$ since $\mathfrak{g}^x = \mathfrak{h}$. Hence τ' is a birational morphism since $B.\mathfrak{h}_{\mathrm{reg}}$ is an open subset of $\mathfrak{b}$.

(ii) Since E is a vector bundle over X and since π is a projective birational morphism, ν is a projective birational morphism. Then τ is a projective birational morphism from $\pi^*(E)$ to $\mathfrak{b}$ by (i). It is B-equivariant since ν and τ' are too. Moreover, $\pi^*(E)$ is a desingularization of $\mathfrak{b}$ since $\pi^*(E)$ is smooth as a vector bundle over a smooth variety. □

Let denote by ψ the canonical projection from $\pi^*(E)$ to Γ. Then, according to the above notations, one has the commutative diagram:

$$\begin{tikzcd} & \pi^*(E) \arrow[r, "\psi"] \arrow[dl, "\tau"'] \arrow[d, "\nu"'] & \Gamma \arrow[d, "\pi"] \\ \mathfrak{b} & E \arrow[l, "\tau'"'] \arrow[r] & X \end{tikzcd}$$

Lemma 7.2. *Let $E^{(k)}$ be the fiber product $\pi^*(E) \times_\psi \cdots \times_\psi \pi^*(E)$ and let τ_k be the canonical morphism from $E^{(k)}$ to $\mathfrak{b}^k$.*

(i) *The vector bundle $E^{(k)}$ over Γ is a vector subbundle of the trivial bundle $\Gamma \times \mathfrak{b}^k$. Moreover, $E^{(k)}$ has dimension $k\ell + n$.*

(ii) *The morphism τ_k is a projective birational morphism from $E^{(k)}$ onto $\mathfrak{X}_{0,k}$. Moreover, $E^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$ in the category of B-varieties.*

Proof. (i) By definition, $E^{(k)}$ is the subvariety of elements $(u, x_1, \ldots, x_k)$ of $\Gamma \times \mathfrak{b}^k$ such that $x_1, \ldots, x_k$ are in $\pi(u)$. Since X is the closure of $B.\mathfrak{h}$, X and Γ have dimension n. Hence $E^{(k)}$ has dimension $k\ell + n$ since $E^{(k)}$ is a vector bundle of rank $k\ell$ over Γ.

(ii) Since Γ is a projective variety, τ_k is a projective morphism and $\tau_k(E^{(k)}) = \mathfrak{X}_{0,k}$ by Lemma 6.1,(i). For $(x_1, \ldots, x_k)$ in $\mathfrak{b}^k_{\mathrm{reg,ss}}$, $\tau_k^{-1}(x_1, \ldots, x_k) = \{(\mathfrak{g}^{x_1}, (x_1, \ldots, x_k))\}$ since $\mathfrak{g}^{x_1}$ is a Cartan subalgebra. Hence τ_k is a birational morphism, whence the assertion since $E^{(k)}$ is a smooth variety as a vector bundle over the smooth variety Γ. □

Let set $\mathfrak{Y} := G \times_B (\Gamma \times \mathfrak{b}^k)$. The canonical projections from $G \times \Gamma \times \mathfrak{b}^k$ to $G \times \Gamma$ and $G \times \mathfrak{b}^k$ define through the quotients morphisms from $\mathfrak{Y}$ to $G \times_B \Gamma$ and $G \times_B \mathfrak{b}^k$. Let denote by ς and ζ these morphisms. Then one has the following diagram:

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "\zeta"] \arrow[d, "\varsigma"'] & G \times_B \mathfrak{b}^k \arrow[d, "\gamma_{\mathrm{n}}"] \\ G \times_B \Gamma & \mathcal{B}_{\mathrm{n}}^{(k)} \end{tikzcd}$$

The map $(g, x) \mapsto (g, \tau_k(x))$ from $G \times E^{(k)}$ to $G \times \mathfrak{b}^k$ defines through the quotient a morphism $\overline{\tau_k}$ from $G \times_B E^{(k)}$ to $\mathfrak{X}_k$.

Proposition 7.3. *Let set $\xi := \gamma_{\mathrm{n}} \circ \overline{\tau_k}$.*

(i) *The variety $G \times_B E^{(k)}$ is a closed subvariety of $\mathfrak{Y}$.*

(ii) *The variety $G \times_B E^{(k)}$ is a vector bundle of rank $k\ell$ over $G \times_B \Gamma$. Moreover, $G \times_B \Gamma$ and $G \times_B E^{(k)}$ are smooth varieties.*

(iii) *The morphism ξ is a projective birational morphism from $G \times_B E^{(k)}$ onto $\mathcal{C}_{\mathrm{n}}^{(k)}$. Moreover $G \times_B E^{(k)}$ is a desingularization of $\mathcal{C}_{\mathrm{n}}^{(k)}$.*

Proof. (i) According to Lemma 7.2,(i), $E^{(k)}$ is a closed subvariety of $\Gamma \times \mathfrak{b}^k$, invariant under the diagonal action of B. Hence $G \times E^{(k)}$ is a closed subvariety of $G \times \Gamma \times \mathfrak{b}^k$, invariant under the action of B, whence the assertion.

(ii) Since $E^{(k)}$ is a B-equivariant vector bundle over Γ, $G \times_B E^{(k)}$ is a G-equivariant vector bundle over $G\times_B \Gamma$. Since $G\times_B\Gamma$ is a fiber bundle over the smooth variety G/B with smooth fibers, $G\times_B\Gamma$ is a smooth variety. As a result, $G \times_B E^{(k)}$ is a smooth variety.

(iii) According to Lemma 7.2,(ii), $\overline{\tau_k}$ is a projective birational morphism from $G\times_B E^{(k)}$ to $\mathfrak{X}_k$. Since $\mathfrak{X}_{0,k}$ is a B-invariant closed subvariety of $\mathfrak{b}^k$, $\mathfrak{X}_k$ is closed in $G\times_B \mathfrak{b}^k$. According to Lemma 6.1,(i), $\gamma(\mathfrak{X}_k) = \mathcal{C}^{(k)}$. Moreover, $\gamma_{\rm n}(\mathfrak{X}_k)$ is a closed subvariety of $\mathcal{B}_{\rm n}^{(k)}$ since $\gamma_{\rm n}$ is a projective morphism by Lemma 1.4. Hence $\gamma_{\rm n}(\mathfrak{X}_k) = \mathcal{C}_{\rm n}^{(k)}$ by Proposition 6.6. For all z in $G.\iota_{\rm n}(\mathfrak{h}^k_{\rm reg})$, $|\gamma_{\rm n}^{-1}(z)| = 1$. Hence the restriction of $\gamma_{\rm n}$ to $\mathfrak{X}_k$ is a birational morphism onto $\mathcal{C}_{\rm n}^{(k)}$ since $G.\iota_{\rm n}(\mathfrak{h}^k_{\rm reg})$ is dense in $\mathcal{C}_{\rm n}^{(k)}$. Moreover, this morphism is projective since $\gamma_{\rm n}$ is projective. As a result, ξ is a projective birational morphism from $G \times_B E^{(k)}$ onto $\mathcal{C}_{\rm n}^{(k)}$ and $G \times_B E^{(k)}$ is a desingularization of $\mathcal{C}_{\rm n}^{(k)}$ by (ii). □

The following corollary results from Lemma 7.2,(ii), Proposition 7.3,(iii) and Lemma 1.1.

Corollary 7.4. *Let $\widetilde{\mathfrak{X}_{0,k}}$ and $\widetilde{\mathcal{C}_{\rm n}^{(k)}}$ be the normalizations of $\mathfrak{X}_{0,k}$ and $\mathcal{C}_{\rm n}^{(k)}$ respectively. Then $\Bbbk[\widetilde{\mathfrak{X}_{0,k}}]$ and $\Bbbk[\widetilde{\mathcal{C}_{\rm n}^{(k)}}]$ are the spaces of global sections of $\mathcal{O}_{E^{(k)}}$ and $\mathcal{O}_{G\times_B E^{(k)}}$ respectively.*

8. Rational singularities

Let $k \geq 2$ be an integer. Let X, X', $X_{\rm n}$, θ_0, $X'_{\rm n}$, Γ, $\pi_{\rm n}$, π, E, $E^{(k)}$, ψ, ν, τ, τ_k be as in Section 7. One has the commutative diagram:

$$\begin{tikzcd} E^{(k)} \arrow[r, "\tau_k"] \arrow[d, "\psi_k"'] & \mathfrak{X}_{0,k} & \\ \Gamma \arrow[r, "\pi_{\rm n}"] \arrow[rr, bend right, "\pi"'] & X_{\rm n} \arrow[r, "\theta_0"] & X \end{tikzcd}$$

with ψ_k the canonical projection from $E^{(k)}$ onto Γ.

8.1. According to the notations of Subsection 5.1, let denote by S_α the closure of $U(\mathfrak{h}_\alpha)$ in $\mathfrak{b}$. For β in Π, let set:

$$\mathfrak{u}_\beta := \bigoplus_{\alpha\in\mathcal{R}_+\setminus\{\beta\}} \mathfrak{g}^\beta \qquad \mathfrak{b}_\beta := \mathfrak{h}_\beta \oplus \mathfrak{u}_\beta$$

Lemma 8.1. *For α in $\mathcal{R}_+$, let $\mathfrak{h}'_\alpha$ be the set of subregular elements belonging to $\mathfrak{h}_\alpha$.*

(i) *For α in $\mathcal{R}_+$, S_α is a subvariety of codimension 2 of $\mathfrak{b}$. Moreover, it is contained in $\mathfrak{b} \setminus \mathfrak{b}_{\rm reg}$.*

(ii) *For β in Π, $S_\beta = \mathfrak{b}_\beta$.*

(iii) *The S_α's, $\alpha \in \mathcal{R}_+$, are the irreducible components of $\mathfrak{b} \setminus \mathfrak{b}_{\rm reg}$.*

Proof. (i) For x in $\mathfrak{h}'_\alpha$, $\mathfrak{b}^x = \mathfrak{h} + \Bbbk x_\alpha$. Hence $U(\mathfrak{h}'_\alpha)$ has dimension $n - 1 + \ell - 1$, whence the assertion since $U(\mathfrak{h}'_\alpha)$ is dense in S_α and since $\mathfrak{h}'_\alpha$ is contained in $\mathfrak{b} \setminus \mathfrak{b}_{\rm reg}$.

(ii) For β in Π, $U(\mathfrak{h}'_\beta)$ is contained in $\mathfrak{b}_\beta$ since $\mathfrak{b}_\beta$ is an ideal of $\mathfrak{b}$, whence the assertion by (i).

(iii) According to (i), it suffices to prove that $\mathfrak{b} \setminus \mathfrak{b}_{\rm reg}$ is the union of the S_α's. Let x be in $\mathfrak{b} \setminus \mathfrak{b}_{\rm reg}$. According to [V72], for some g in G and for some β in Π, x is in $g(\mathfrak{b}_\beta)$. Since $\mathfrak{b}_\beta$ is an ideal of $\mathfrak{b}$, by

Bruhat's decomposition of G, for some b in B and for some w in $W(\mathcal{R})$, $b^{-1}(x)$ is in $w(\mathfrak{b}_\beta) \cap \mathfrak{b}$. By definition,

$$w(\mathfrak{b}_\beta) = w(\mathfrak{h}_\beta) \oplus w(\mathfrak{u}_\beta) = \mathfrak{h}_{w(\beta)} \oplus \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} \mathfrak{g}^{w(\alpha)}$$

So,

$$w(\mathfrak{b}_\beta) \cap \mathfrak{b} = \mathfrak{h}_{w(\beta)} \oplus \mathfrak{u}_0 \text{ with } \mathfrak{u}_0 := \bigoplus_{\substack{\alpha \in \mathcal{R}_+ \setminus \{\beta\} \\ w(\alpha) \in \mathcal{R}_+}} \mathfrak{g}^{w(\alpha)}$$

The subspace $\mathfrak{u}_0$ of $\mathfrak{u}$ is a subalgebra, not containing $\mathfrak{g}^{w(\beta)}$. Then, denoting by U_0 the closed subgroup of U whose Lie algebra is $\operatorname{ad} \mathfrak{u}_0$,

$$\overline{U_0(\mathfrak{h}_{w(\beta)})} = w(\mathfrak{b}_\beta) \cap \mathfrak{b}$$

since the left hand side is contained in the right hand side and has the same dimension. As a result, x is in $S_{w(\beta)}$ since $S_{w(\beta)}$ is B-invariant, whence the assertion. □

Let $\mathfrak{g}'_{\rm reg}$ be the set of regular elements x such that $x_{\rm s}$ is regular or subregular and let set $\mathfrak{b}'_{\rm reg} := \mathfrak{g}'_{\rm reg} \cap \mathfrak{b}$.

Lemma 8.2. (i) *The subset $\mathfrak{b}'_{\rm reg}$ of $\mathfrak{b}$ is a big open subset of $\mathfrak{b}$.*

(ii) *The subset $\mathfrak{g}'_{\rm reg}$ of $\mathfrak{g}$ is a big open subset of $\mathfrak{g}$.*

Proof. Let x be in $\mathfrak{g}'_{\rm reg} \setminus \mathfrak{g}_{\rm reg,ss}$. Let W be the set of elements y of $\mathfrak{g}^{x_{\rm s}}$ such that the restriction of $\operatorname{ad} y$ to $[x_{\rm s}, \mathfrak{g}]$ is injective. Then W is an open subset of $\mathfrak{g}^{x_{\rm s}}$, containing x, and the map

$$G \times W \longrightarrow \mathfrak{g} \qquad (g, y) \longmapsto g(y)$$

is a submersion. Let $\mathfrak{z}$ be the centre of $\mathfrak{g}^{x_{\rm s}}$ and let set $\mathfrak{z}' := W \cap \mathfrak{z}$. For some open subset W' of W, containing x, for all y in W', the component of y on $\mathfrak{z}$ is in $\mathfrak{z}'$. Since $[\mathfrak{g}^{x_{\rm s}}, \mathfrak{g}^{x_{\rm s}}]$ is a simple algebra of dimension 3, $W' \cap \mathfrak{g}_{\rm reg}$ is contained in $\mathfrak{g}'_{\rm reg}$ and $G(W' \cap \mathfrak{g}_{\rm reg})$ is an open set, contained in $\mathfrak{g}'_{\rm reg}$ and containing x. As a result, $\mathfrak{g}'_{\rm reg}$ is an open subset of $\mathfrak{g}$ and $\mathfrak{b}'_{\rm reg}$ is an open subset of $\mathfrak{b}$.

(i) Let suppose that $\mathfrak{b} \setminus \mathfrak{b}'_{\rm reg}$ has an irreducible component Σ of codimension 1 in $\mathfrak{b}$. One expects a contradiction. Since Σ is invariant under B, $\Sigma \cap \mathfrak{h}$ is the image of Σ by the projection $x \mapsto \overline{x}$ by Lemma 1.5. Since Σ has codimension 1 in $\mathfrak{b}$, $\Sigma \cap \mathfrak{h} = \mathfrak{h}$ or $\Sigma = \Sigma \cap \mathfrak{h} + \mathfrak{u}$. Since Σ does not contain regular semisimple element, $\Sigma \cap \mathfrak{h}$ is an irreducible subset of codimension 1 of $\mathfrak{h}$, not containing regular semisimple elements. Hence $\Sigma \cap \mathfrak{h} = \mathfrak{h}_\alpha$ for some positive root and $\Sigma \cap (\mathfrak{h}'_\alpha + \mathfrak{g}^\alpha) \cap \mathfrak{g}_{\rm reg}$ is not empty, whence the contradiction.

(ii) Since $\mathfrak{b} \setminus \mathfrak{b}'_{\rm reg}$ is invariant under B, $\mathfrak{g} \setminus \mathfrak{g}'_{\rm reg} = G(\mathfrak{b} \setminus \mathfrak{b}'_{\rm reg})$ and

$$\dim \mathfrak{g} \setminus \mathfrak{g}'_{\rm reg} \leq n + \dim \mathfrak{b} \setminus \mathfrak{b}'_{\rm reg}$$

whence the assertion by (i). □

Setting $\mathfrak{b}_{\rm reg,0} := \mathfrak{b}_{\rm reg}$ and $\mathfrak{b}_{\rm reg,1} := \mathfrak{b}'_{\rm reg}$, let $V_{k,j}$ be the subset of elements x of $\mathfrak{X}_{0,k}$ such that $P_x \cap \mathfrak{b}_{{\rm reg},j}$ is not empty for $j = 0, 1$.

Proposition 8.3. *For $j = 0, 1$, let $V'_{k,j}$ be the subset of elements $x = (x_1, \ldots, x_k)$ of $\mathfrak{X}_{0,k}$ such that x_1 is in $\mathfrak{b}_{{\rm reg},j}$.*

(i) *For $j = 0, 1$, $V'_{k,j}$ is a smooth open subset of $\mathfrak{X}_{0,k}$.*

(ii) *For $j = 0, 1$, $V_{k,j}$ is a smooth open subset of $\mathfrak{X}_{0,k}$.*

(iii) *For $j = 0, 1$, $V_{k,j}$ is a big open subset of $\mathfrak{X}_{0,k}$.*

Proof. (i) By definition, $V'_{k,j}$ is the intersection of $\mathfrak{X}_{0,k}$ and the open subset $\mathfrak{b}_{\mathrm{reg},j} \times \mathfrak{b}^{k-1}$ of $\mathfrak{b}^k$. Hence $V'_{k,j}$ is an open subset of $\mathfrak{X}_{0,k}$. For x_1 in $\mathfrak{b}_{\mathrm{reg},0}$, $(x_1,\dots,x_k)$ is in $V'_{k,0}$ if and only if $x_2,\dots,x_k$ are in $\mathfrak{g}^{x_1}$ by Corollary 5.3,(ii) and Lemma 7.2,(ii) since $\mathfrak{g}^{x_1}$ is in X. According to [Ko63, Theorem 9], for x in $\mathfrak{b}_{\mathrm{reg}}$, $\varepsilon_1(x),\dots,\varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$. Hence the map

$$\begin{gathered} \mathfrak{b}_{\mathrm{reg}} \times \mathrm{M}_{k-1,\ell}(\Bbbk) \xrightarrow{\theta} V'_{k,0} \\ (x, (a_{i,j}, 1 \leq i \leq k-1, 1 \leq j \leq \ell)) \longmapsto (x, \textstyle\sum_{j=1}^{\ell} a_{1,j}\varepsilon_j(x), \dots, \sum_{j=1}^{\ell} a_{k-1,j}\varepsilon_j(x)) \end{gathered}$$

is a bijective morphism. The open subset $\mathfrak{b}_{\mathrm{reg}}$ has a cover by open subsets V such that for some $e_1,\dots,e_n$ in $\mathfrak{b}$, $\varepsilon_1(x),\dots,\varepsilon_\ell(x), e_1,\dots,e_n$ is a basis for all x in V. Then there exist regular functions $\varphi_1,\dots,\varphi_\ell$ on $V \times \mathfrak{b}$ such that

$$v - \sum_{j=1}^{\ell} \varphi_j(x,v)\varepsilon_j(x) \in \mathrm{span}(e_1,\dots,e_n)$$

for all (x,v) in $V \times \mathfrak{b}$, so that the restriction of θ to $V \times \mathrm{M}_{k-1,\ell}(\Bbbk)$ is an isomorphism onto $\mathfrak{X}_{0,k} \cap V \times \mathfrak{b}^{k-1}$ whose inverse is

$$(x_1,\dots,x_k) \longmapsto (x_1, ((\varphi_1(x_1,x_i),\dots,\varphi_\ell(x_1,x_i)), i = 2,\dots,k))$$

As a result, θ is an isomorphism and $V'_{k,0}$ is a smooth variety, whence the assertion since $V'_{k,1}$ is an open subset of $V'_{k,0}$.

(ii) The subvariety $\mathfrak{X}_{0,k}$ of $\mathfrak{b}^k$ is invariant under the natural action of $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{b}^k$ and $V_{k,j} = \mathrm{GL}_k(\Bbbk).V'_{k,j}$ by Lemma 1.6, whence the assertion by (i).

(iii) Since $V_{k,1}$ is contained in $V_{k,0}$, it suffices to prove the assertion for $j = 1$. Let suppose that $\mathfrak{X}_{0,k} \setminus V_{k,1}$ has an irreducible component Σ of codimension 1. One expects a contradiction. Since $\mathfrak{X}_{0,k}$ and $V_{k,1}$ are invariant under B and $\mathrm{GL}_k(\Bbbk)$, it is so for Σ. Since Σ has codimension 1 in $\mathfrak{X}_{0,k}$, $\tau_k^{-1}(\Sigma)$ has codimension 1 in $E^{(k)}$. Let Σ_0 be an irreducible component of codimension 1 of $\tau_k^{-1}(\Sigma)$ and let set $T := \pi\circ\psi_k(\Sigma_0)$. Since Σ is invariant under $\mathrm{GL}_k(\Bbbk)$, Σ_0 is invariant under the action of $\mathrm{GL}_k(\Bbbk)$ so that the intersection of $(\pi\circ\psi_k)^{-1}(T)$ and the null section of $E^{(k)}$ is contained in Σ_0. So, T is a closed irreducible subset of X. Moreover, T is strictly contained in X. Indeed, if it is not so, for all u in $U.\mathfrak{h}$, $\{u\} \times u^k \cap \Sigma_0$ has dimension at most $k(l-1)$ since Σ_0 is invariant under $\mathfrak{S}_k$. Then T has codimension 1 in X and $\Sigma_0 = (\pi\circ\psi_k)^{-1}(T)$. According to Theorem 5.13,(ii), for some u in T, $u \cap \mathfrak{b}_{\mathrm{reg},1}$ is not empty, whence the contradiction since for all x in Σ, $P_x \cap \mathfrak{b}_{\mathrm{reg},1}$ is empty and since u^k is contained in Σ for all u in T. □

Let $\widetilde{\mathfrak{X}_{0,k}}$ be the normalization of $\mathfrak{X}_{0,k}$ and let λ_k be the normalization morphism whence a commutative diagram

$$\begin{tikzcd} E^{(k)} \arrow[r, "\widetilde{\tau_k}"] \arrow[dr, "\tau_k"'] & \widetilde{\mathfrak{X}_{0,k}} \arrow[d, "\lambda_k"] \\ & \mathfrak{X}_{0,k} \end{tikzcd}$$

since $(E^{(k)}, \tau_k)$ is a desingularization of $\mathfrak{X}_{0,k}$.

Corollary 8.4. *For $j = 0, 1$, $\lambda_k^{-1}(V_{k,j})$ is a smooth big open susbset of $\widetilde{\mathfrak{X}_{0,k}}$ and the restriction of $\widetilde{\tau_k}$ to $\tau_k^{-1}(V_{k,j})$ is an isomorphism onto $\lambda_k^{-1}(V_{k,j})$.*

Proof. According to Proposition 8.3, $V_{k,j}$ is a smooth big open subset of $\mathfrak{X}_{0,k}$. Hence the restriction of λ_k to $\lambda_k^{-1}(V_{k,j})$ is an isomorphism onto $V_{k,j}$. For all x in $V_{k,j}$, $\tau_k^{-1}(x) = (u, x)$ with u equal to the centralizer of a regular element contained in P_x. Hence, by Zariski Main Theorem [Mu88, /S 9], the restriction of τ_k to $\tau_k^{-1}(V_{k,j})$ is an isomorphism onto $V_{k,j}$ since $V_{k,j}$ is smooth, whence the corollary. □

8.2. By definition, the restriction of $\pi_{\rm n}$ to $\pi_{\rm n}^{-1}(X'_{\rm n})$ is an isomorphism onto $X'_{\rm n}$. Let identify $\pi_{\rm n}^{-1}(X'_{\rm n})$ and $X'_{\rm n}$ by $\pi_{\rm n}$. Let denote by E_k the restriction of $E^{(k)}$ to $X'_{\rm n}$. According to Proposition 5.20,(ii), θ_0 is a homeomorphism from $\theta_0^{-1}(X')$ to X'. Moreover, $U.\mathfrak{h}$ identifies with an open subset of $X'_{\rm n}$ since it is a smooth open subset of X'.

Lemma 8.5. *Let set $E_{\rm n} := \theta_0^*(E)$ and let denote by $\nu_{\rm n}$ the canonical morphism from $E_{\rm n}$ to E.*

(i) *There exists a well defined projective birational morphism $\tau_{\rm n}$ from $\pi^*(E)$ to $E_{\rm n}$ such that $\nu = \nu_{\rm n}\circ\tau_{\rm n}$. Moreover, $E_{\rm n}$ is normal.*

(ii) *The $\mathcal{O}_{\pi^*(E)}$-module $\Omega_{\pi^*(E)}$ is free.*

(iii) *The variety $E_{\rm n}$ is Gorenstein and has rational singularities.*

Proof. (i) Since $E_{\rm n}$ is a vectore bundle over $X_{\rm n}$, $E_{\rm n}$ is a normal variety. Moreover, it is the normalization of E and $\nu_{\rm n}$ is the normalization morphism, whence the assertion by Lemma 7.1,(ii).

(ii) Let ω be a volume form on $\mathfrak{b}$. According to Lemma 7.1,(ii), $\tau^*(\omega)$ is a global section of $\Omega_{\pi^*(E)}$, without zero, whence the assertion since $\Omega_{\pi^*(E)}$ is locally free of rank 1.

(iii) According to (ii), $\mathcal{O}_{\pi^*(E)}$ is isomorphic to $\Omega_{\pi^*(E)}$. So, by Grauert-Riemenschneider Theorem [GR70], $\mathrm{R}^i(\tau_{\rm n})_*(\mathcal{O}_{\pi^*(E)}) = 0$ for $i > 0$. Hence $E_{\rm n}$ has rational singularities by (i). Moreover, $(\tau_{\rm n})_*(\Omega_{\pi^*(E)})$ is free of rank 1 by (ii). In other words, the canonical module of $E_{\rm n}$ is isomorphic to $\mathcal{O}_{E_{\rm n}}$, that is $E_{\rm n}$ is Gorenstein. □

Let $\rho_{\rm n}$ be the canonical projection from $E_{\rm n}$ to $X_{\rm n}$ and let set $E_{\rm n}^{(k)} := \underbrace{E_{\rm n}\times_{\rho_{\rm n}}\cdots\times_{\rho_{\rm n}} E_{\rm n}}_{k \text{ factors}}$.

Corollary 8.6. (i) *The variety $E^{(k)}$ is a desingularization of $E_{\rm n}^{(k)}$.*

(ii) *The variety $E_{\rm n}^{(k)}$ is Gorenstein and has rational singularities.*

Proof. (i) Let ρ be the canonical projection from E to X and let set $\widetilde{E}^{(k)} := \underbrace{E\times_{\rho}\cdots\times_{\rho} E}_{k \text{ factors}}$. Since $E_{\rm n}^{(k)}$ is a vector bundle over the normal variety $X_{\rm n}$, $E_{\rm n}^{(k)}$ is a normal variety. Moreover, it is the normalization of $\widetilde{E}^{(k)}$ since $X_{\rm n}$ is the normalization of X, whence a commutative diagram

$$\begin{tikzcd} E^{(k)} \arrow[r] \arrow[dr] & E_{\rm n}^{(k)} \arrow[d] \\ & \widetilde{E}^{(k)} \end{tikzcd}$$

According to Lemma 7.2,(ii), the diagonal arrow is a birational projective morphism. Hence the horizontal arrow is birational and projective.

(ii) The variety $E_{\rm n}^{(k)}$ is a vector bundle over $E_{\rm n}$. So, by Lemma 8.5,(iii), $E_{\rm n}^{(k)}$ is Gorenstein and has rational singularities. □

Theorem 8.7. *The normalization $\widetilde{\mathfrak{X}_{0,k}}$ of $\mathfrak{X}_{0,k}$ has rational singularities.*

Proof. By definition, the morphism $\widetilde{\tau_k}$ from $E^{(k)}$ to $\widetilde{\mathfrak{X}_{0,k}}$ factorizes through the morphism $E^{(k)} \rightarrow E_{\mathrm{n}}^{(k)}$ so that there is a commutative diagram

$$\begin{tikzcd} E^{(k)} \arrow[r] \arrow[dr, "\widetilde{\tau_k}"'] & E_{\mathrm{n}}^{(k)} \arrow[d] \\ & \widetilde{\mathfrak{X}_{0,k}} \end{tikzcd}$$

Moroeover, according to Lemma 7.2,(ii) and Corollary 8.6,(i), all the arrows are projective and birational. According to the previous identifications, E_k is a smooth big open subset of $E_{\mathrm{n}}^{(k)}$ since X'_{n} is a smooth big open subset of X_{n}. According to Corollary 8.4, the open subset $V_{k,1}$ of $\mathfrak{X}_{0,k}$ identifies with its inverse images in $\widetilde{\mathfrak{X}_{0,k}}$ and E_k. Moreover, $V_{k,1}$ is a big open subset of $\widetilde{\mathfrak{X}_{0,k}}$. For all Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$, contained in $\mathfrak{b}$, $\mathfrak{c}^k \setminus V_{k,1}$ is contained in $(\mathfrak{c} \setminus \mathfrak{g}_{\mathrm{reg}})^k$ so that it has codimension at least 2 in $\mathfrak{c}^k$ since $k \geq 2$. As a result, $V_{k,1}$ is a big open subset of $E_{\mathrm{n}}^{(k)}$ since for all u in $X' \setminus U.\mathfrak{h}$, u^k is not contained in $V_{k,1}$. Then, according to Corollary 8.6 and Proposition C.2, with $Y = E_{\mathrm{n}}^{(k)}$, $\widetilde{\mathfrak{X}_{0,k}}$ has rational singularities. □

8.3. Let denote by E^* the dual of the vector bundle $\pi^*(E)$ over Γ.

Lemma 8.8. *Let $\mathcal{E}^*$ be the sheaf of local sections of E^*. For $i > 0$ and for $j \geq 0$, $\mathrm{H}^i(\Gamma, \mathrm{S}^j(\mathcal{E}^*)) = 0$.*

Proof. Since ψ is the canonical projection from $\pi^*(E)$ to Γ, $\mathcal{O}_{\pi^*(E)}$ equals $\psi^*(\mathrm{S}(\mathcal{E}^*))$ so that

$$(\psi)_*(\mathcal{O}_{\pi^*(E)}) = \mathrm{S}(\mathcal{E}^*)$$

As a result, for $i \geq 0$,

$$\mathrm{H}^i(\pi^*(E), \mathcal{O}_{\pi^*(E)}) = \mathrm{H}^i(\Gamma, \mathrm{S}(\mathcal{E}^*)) = \bigoplus_{j \in \mathbb{N}} \mathrm{H}^i(\Gamma, \mathrm{S}^j(\mathcal{E}^*))$$

According to Lemma 7.1,(ii), $\pi^*(E)$ is a desingularization of the smooth variety $\mathfrak{b}$. Hence by [El78],

$$\mathrm{H}^i(\pi^*(E), \mathcal{O}_{\pi^*(E)}) = 0$$

for $i > 0$, whence

$$\mathrm{H}^i(\Gamma, \mathrm{S}^j(\mathcal{E}^*)) = 0$$

for $i > 0$ and $j \geq 0$. □

According to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by the Killing form, $\mathfrak{b}_-$ identifies with $\mathfrak{b}^*$. Let denote by E_- the orthogonal complement of $\pi^*(E)$ in $\Gamma \times \mathfrak{b}_-$ so that E_- is a vector bundle of rank n over Γ. Let $\mathcal{E}_-$ be the sheaf of local sections of E_-.

Corollary 8.9. *Let $\mathfrak{J}_0$ be the ideal of $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ generated by $\mathcal{E}_-$. Then, for $i \geq 0$, $\mathrm{H}^i(\Gamma, \mathfrak{J}_0) = 0$ and $\mathrm{H}^i(\Gamma, \mathcal{E}_-) = 0$.*

Proof. Since E_- is the orthogonal complement of $\pi^*(E)$ in $\Gamma \times \mathfrak{b}_-$, $\mathfrak{J}_0$ is the ideal of definition of $\pi^*(E)$ in $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ whence a short exact sequence

$$0 \longrightarrow \mathfrak{J}_0 \longrightarrow \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-) \longrightarrow \mathrm{S}(\mathcal{E}^*) \longrightarrow 0$$

and whence a cohomology long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(\Gamma, \mathrm{S}(\mathcal{E}^*)) \longrightarrow \mathrm{H}^{i+1}(\Gamma, \mathcal{J}_0) \longrightarrow \mathrm{H}^{i+1}(\Gamma, \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)) \longrightarrow \cdots$$

Then, by Lemma 8.8, from the equality

$$\mathrm{H}^i(\Gamma, \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)) = \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathrm{H}^i(\Gamma, \mathcal{O}_\Gamma)$$

for all i, one deduces $\mathrm{H}^i(\Gamma, \mathcal{J}_0) = 0$ for $i \geq 2$. Moreover, since Γ is an irreducible projective variety, $\mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma) = \Bbbk$ and since $\pi^*(E)$ is a desingularization of $\mathfrak{b}$, $\mathrm{H}^0(\Gamma, \mathrm{S}(\mathcal{E}^*)) = \mathrm{S}(\mathfrak{b}_-)$ so that the map

$$\mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)) \longrightarrow \mathrm{H}^0(\Gamma, \mathrm{S}(\mathcal{E}^*))$$

is an isomorphism. Hence $\mathrm{H}^i(\Gamma, \mathcal{J}_0) = 0$ for $i = 0, 1$. The gradation on $\mathrm{S}(\mathfrak{b}_-)$ induces a gradation on $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ so that $\mathcal{J}_0$ is a graded ideal. Since $\mathcal{E}_-$ is the subsheaf of local sections of degree 1 of $\mathcal{J}_0$, it is a direct factor of $\mathcal{J}_0$, whence the corollary. □

Proposition 8.10. *Let l, m be nonnegative integers.*

(i) *For all positive integer i,* $\mathrm{H}^i(\Gamma, (\mathcal{E}^*)^{\otimes m}) = 0$.

(ii) *For all positive integer i,*

$$\mathrm{H}^{i+l}(\Gamma, \mathcal{E}_-^{\otimes l} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes m}) = 0$$

Proof. (i) According to Lemma 8.8, one can suppose $m > 1$. Since E^* is the dual of the vector bundle $\pi^*(E)$ over Γ, the fiber product $E^*_m := E^* \times_\psi \cdots \times_\psi E^*$ is the dual of the vector bundle $E^{(m)}$ over Γ. Let ψ_m be the canonical projection from E^*_m to Γ and let $\mathcal{E}^*_m$ be the sheaf of local sections of $E^{(*)}_m$. Then $\mathcal{O}_{E^{(m)}}$ equals $\psi^*_m(\mathrm{S}(\mathcal{E}^*_m))$ and since $E^{(m)}$ is a vector bundle over Γ, for all nonnegative integer i,

$$\mathrm{H}^i(E^{(m)}, \mathcal{O}_{E^{(m)}}) = \mathrm{H}^i(\Gamma, \mathrm{S}(\mathcal{E}^*_m)) = \bigoplus_{q \in \mathbb{N}} \mathrm{H}^i(\Gamma, \mathrm{S}^q(\mathcal{E}^*_m))$$

According to Theorem 8.7, for $i > 0$, the left hand side equals 0 since $E^{(m)}$ is a desingularization of $\widetilde{\mathfrak{X}_{0,m}}$ by Lemma 7.2,(iv). As a result, for $i > 0$,

$$\mathrm{H}^i(\Gamma, \mathrm{S}^m(\mathcal{E}^*_m))) = 0$$

The decomposition of $\mathcal{E}^*_m$ as a direct sum of m copies isomorphic to $\mathcal{E}^*$ induces a multigradation of $\mathrm{S}(\mathcal{E}^*)$. Denoting by $\mathcal{S}_{j_1,\ldots,j_m}$ the subsheaf of multidegree $(j_1, \ldots, j_m)$, one has

$$\mathrm{S}^m(\mathcal{E}^*_m) = \bigoplus_{\substack{(j_1,\ldots,j_m) \in \mathbb{N}^m \\ j_1+\cdots+j_m=m}} \mathcal{S}_{j_1,\ldots,j_m} \text{ and } \mathcal{S}_{1,\ldots,1} = (\mathcal{E}^*)^{\otimes m}$$

Hence for $i > 0$,

$$0 = \mathrm{H}^i(\Gamma, \mathrm{S}^m(\mathcal{S}^*_m)) = \bigoplus_{\substack{(j_1,\ldots,j_m) \in \mathbb{N}^m \\ j_1+\cdots+j_m=m}} \mathrm{H}^i(\Gamma, \mathcal{S}_{j_1,\ldots,j_m})$$

whence the assertion.

(ii) Let m be a nonnegative integer. Let prove by induction on j that for $i > 0$ and for $l \geq j$,

$$\mathrm{H}^{i+j}(\Gamma, \mathcal{E}_-^{\otimes j} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j)}) = 0 \tag{6}$$

By (i) it is true for $j = 0$. Let suppose $j > 0$ and (6) true for $j-1$ and for all $l \geq j-1$. From the short exact sequence of $\mathcal{O}_\Gamma$-modules

$$0 \longrightarrow \mathcal{E}_- \longrightarrow \mathcal{O}_\Gamma \otimes_\Bbbk \mathfrak{b}_- \longrightarrow \mathcal{E}^* \longrightarrow 0$$

one deduces the short exact sequence of $\mathcal{O}_\Gamma$-modules

$$0 \longrightarrow \mathcal{E}_-^{\otimes j} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j)} \longrightarrow \mathfrak{b}_- \otimes_\Bbbk \mathcal{E}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j)} \longrightarrow \mathcal{E}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j+1)} \longrightarrow 0$$

From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

$$\begin{aligned} \mathrm{H}^{i+j-1}(\Gamma, \mathcal{E}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j+1)}) \longrightarrow \mathrm{H}^{i+j}(\Gamma, \mathcal{E}_-^{\otimes j} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j)}) \\ \longrightarrow \mathrm{H}^{i+j}(\Gamma, \mathfrak{b}_- \otimes_\Bbbk \mathcal{E}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j)}) \end{aligned}$$

for all positive integer i. By induction hypothesis, the first term equals 0 for all $i > 0$. Since

$$\mathrm{H}^{i+j}(\Gamma, \mathfrak{b}_- \otimes_\Bbbk \mathcal{E}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j)}) = \mathfrak{b}_- \otimes_\Bbbk \mathrm{H}^{i+j}(\Gamma, \mathcal{E}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{E}^*)^{\otimes(m+l-j)})$$

the last term of the last exact sequence equals 0 by induction hypothesis again, whence Equality (6) and whence the assertion for $j = l$. □

The following corollary results from Proposition 8.10,(ii) and Proposition B.2.

Corollary 8.11. *For m positive integer and for $l = (l_1, \dots, l_m)$ in $\mathbb{N}^m$,*

$$\mathrm{H}^{i+|l|}(\Gamma, \textstyle\bigwedge^{l_1}(\mathcal{E}_-) \otimes_{\mathcal{O}_\Gamma} \cdots \otimes_{\mathcal{O}_\Gamma} \bigwedge^{l_m}(\mathcal{E}_-)) = 0$$

for all positive integer i.

8.4. By definition, $E^{(k)}$ is a closed subvariety of $\Gamma \times \mathfrak{b}^k$. Let denote by ϱ the canonical projection from $\Gamma \times \mathfrak{b}^k$ to Γ, whence the diagram

$$\begin{array}{ccc} E^{(k)} & \hookrightarrow & \Gamma \times \mathfrak{b}^k \\ & {\scriptstyle \psi_k}\searrow & \downarrow{\scriptstyle \varrho} \\ & & \Gamma \end{array}$$

For $j = 1, \dots, k$, let denote by $\mathfrak{S}_{j,k}$ the set of injections from $\{1, \dots, j\}$ to $\{1, \dots, k\}$ and for σ in $\mathfrak{S}_{j,k}$, let set:

$$\mathcal{K}_\sigma := \mathcal{M}_1 \otimes_{\mathcal{O}_\Gamma} \cdots \otimes_{\mathcal{O}_\Gamma} \mathcal{M}_k \text{ with } \mathcal{M}_i := \begin{cases} \mathcal{O}_\Gamma \otimes_\Bbbk \mathrm{S}(\mathfrak{b}_-) & \text{if} \quad i \notin \sigma(\{1, \dots, j\}) \\ \mathcal{J}_0 & \text{if} \quad i \in \sigma(\{1, \dots, j\}) \end{cases}$$

For j in $\{1, \dots, k\}$, the direct sum of the $\mathcal{K}_\sigma$'s is denoted by $\mathcal{J}_{j,k}$ and for σ in $\mathfrak{S}_{1,k}$, $\mathcal{K}_\sigma$ is also denoted by $\mathcal{K}_{\sigma(1),k}$.

Lemma 8.12. *Let $\mathcal{J}$ be the ideal of definition of $E^{(k)}$ in $\mathcal{O}_{\Gamma\times\mathfrak{b}^k}$.*

(i) *The ideal $\varrho_*(\mathcal{J})$ of $\mathcal{O}_\Gamma \otimes_\Bbbk \mathrm{S}(\mathfrak{b}_-^k)$ is the sum of $\mathcal{K}_{1,k}, \dots, \mathcal{K}_{k,k}$.*

(ii) *There is an exact sequence of $\mathcal{O}_\Gamma$-modules*

$$0 \longrightarrow \mathcal{J}_{k,k} \longrightarrow \mathcal{J}_{k-1,k} \longrightarrow \cdots \longrightarrow \mathcal{J}_{1,k} \longrightarrow \varrho_*(\mathcal{J}) \longrightarrow 0$$

(iii) *For $i > 0$, $\mathrm{H}^i(\Gamma \times \mathfrak{b}^k, \mathcal{J}) = 0$ if $\mathrm{H}^{i+j}(\Gamma, \mathcal{J}_0^{\otimes j}) = 0$ for $j = 1, \dots, k$.*

Proof. (i) Let $\mathcal{J}_k$ be the sum of $\mathcal{K}_{1,k},\ldots,\mathcal{K}_{k,k}$. Since $\mathcal{J}_0$ is the ideal of $\mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-)$ generated by $\mathcal{E}_-$, $\mathcal{J}_k$ is a prime ideal of $\mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-^k)$. Moreover, $\mathcal{E}_-$ is the sheaf of local sections of the orthogonal complement of E in $\Gamma\times\mathfrak{b}_-$. Hence $\mathcal{J}_k$ is the ideal of definition of $E^{(k)}$ in $\mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-^k)$, whence the assertion.

(ii) For a a local section of $\mathcal{J}_{j,k}$ and for σ in $\mathfrak{S}_{j,k}$, let denote by $a_{\sigma(1),\ldots,\sigma(j)}$ the component of a on $\mathcal{K}_\sigma$. Let d be the map $\mathcal{J}_{j,k}\rightarrow\mathcal{J}_{j-1,k}$ such that

$$\mathrm{d}a_{i_1,\ldots,i_j}=\sum_{l=1}^{j}(-1)^{l+1}a_{i_1,\ldots,i_{l-1},i_{l+1},\ldots,i_j}$$

Then by (i), one has an augmented complex

$$0\longrightarrow\mathcal{J}_{k,k}\xrightarrow{\mathrm{d}}\mathcal{J}_{k-1,k}\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\mathcal{J}_{1,k}\longrightarrow\varrho_*(\mathcal{J})\longrightarrow 0$$

Let J the the subbundle of the trivial bundle $\Gamma\times\mathrm{S}(\mathfrak{b}_-)$ such that the fiber at x is the ideal of $\mathrm{S}(\mathfrak{b}_-)$ generated by the fiber $E_{-,x}$ of E_- at x. Then $\mathcal{J}_0$ is the sheaf of local sections of J and the above augmented complex is the sheaf of local sections of the augmented complex of vector bundles over Γ,

$$0\longrightarrow C_k^{(k)}(\Gamma\times\mathrm{S}(\mathfrak{b}_-),J)\longrightarrow\cdots\longrightarrow C_1^{(k)}(\Gamma\times\mathrm{S}(\mathfrak{b}_-),J)\rightarrow J\longrightarrow 0$$

According to Lemma B.3 and Remark B.4, this complex is acyclic, whence the assertion by Nakayama Lemma since J and $\mathrm{S}(\mathfrak{b}_-)$ are graded.

(iii) Let i be a positive integer such that $\mathrm{H}^{i+j}(\Gamma,\mathcal{J}_0^{\otimes j})=0$ for $j=1,\ldots,k$. Then for $j=1,\ldots,k$ and for σ in $\mathfrak{S}_{j,k}$, $\mathrm{H}^{i+j}(\Gamma,\mathcal{K}_\sigma)=0$ since $\mathcal{K}_\sigma$ is isomorphic to a sum of copies of $\mathcal{J}_0^{\otimes j}$. Moreover, $\mathrm{H}^i(\Gamma,\mathcal{K}_{l,k})=0$ for $l=1,\ldots,k$ since $\mathrm{H}^i(\Gamma,\mathcal{J}_0)=0$ by Corollary 8.9. Hence by (ii), since $\mathrm{H}^\bullet$ is an exact δ-functor, $\mathrm{H}^i(\Gamma,\varrho_*(\mathcal{J}))=0$, whence the assertion since ϱ is an affine morphism. □

8.5. For m positive integer, for j nonnegative integer and for $l=(l_1,\ldots,l_m)$ in $\mathbb{N}^m$, let set:

$$\mathcal{M}_{j,l}:=\mathcal{J}_0^{\otimes j}\otimes_{\mathcal{O}_\Gamma}\textstyle\bigwedge^{l_1}(\mathcal{E}_-)\otimes_{\mathcal{O}_\Gamma}\cdots\otimes_{\mathcal{O}_\Gamma}\bigwedge^{l_m}(\mathcal{E}_-)$$

Lemma 8.13. *Let j,m be positive integers and let l be in $\mathbb{N}^m$.*

(i) *The $\mathcal{O}_\Gamma$-module $\mathcal{J}_0$ is locally free.*

(ii) *There is an exact sequence*

$$\begin{aligned}0\longrightarrow\mathrm{S}(\mathfrak{b}_-)\otimes_{\Bbbk}\mathcal{M}_{j-1,(n,l)}\longrightarrow\mathrm{S}(\mathfrak{b}_-)\otimes_{\Bbbk}\mathcal{M}_{j-1,(n-1,l)}\longrightarrow\cdots\\ \longrightarrow\mathrm{S}(\mathfrak{b}_-)\otimes_{\Bbbk}\mathcal{M}_{j-1,(1,l)}\longrightarrow\mathcal{M}_{j,l}\longrightarrow 0\end{aligned}$$

(iii) *For $i>0$, $\mathrm{H}^{i+j+|l|}(\Gamma,\mathcal{M}_{j,l})=0$.*

Proof. (i) Let x be in Γ and let $E_{-,x}$ be the fiber at x of the vector bundle E_- over Γ. Then $E_{-,x}$ is a subspace of dimension n of $\mathfrak{b}_-$. Let M be a complement of $E_{-,x}$ in $\mathfrak{b}_-$. Since the map $y\mapsto E_{-,y}$ is a regular map from Γ to $\mathrm{Gr}_n(\mathfrak{b}_-)$, for all y in an open neighborhood V of x in Γ,

$$\mathfrak{b}_-=E_{-,x}\oplus M$$

Denoting by $\mathcal{E}_{-,V}$ the restriction of $\mathcal{E}_-$ to V, one has

$$\mathcal{O}_V\otimes_{\Bbbk}\mathfrak{b}_-=\mathcal{E}_{-,V}\oplus\mathcal{O}_V\otimes_{\Bbbk}M$$

so that

$$\mathcal{O}_V \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-) = \mathrm{S}(\mathcal{E}_{-,V}) \otimes_{\Bbbk} \mathrm{S}(M)$$

whence

$$\mathcal{J}_0\,|_V = \mathrm{S}_+(\mathcal{E}_{-,V}) \otimes_{\Bbbk} \mathrm{S}(M)$$

As a result, $\mathcal{J}_0$ is locally free since $\mathcal{E}_-$ is locally free.

(ii) Since $\mathcal{J}_0$ is the ideal of $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ generated by the locally free module $\mathcal{E}_-$ of rank n and since $\mathcal{E}_-$ is locally generated by a regular sequence of the algebra $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$, having n elements, one has an exact Koszul complex

$$0 \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \textstyle\bigwedge^n(\mathcal{E}_-) \longrightarrow \cdots \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{E}_- \longrightarrow \mathcal{J}_0 \longrightarrow 0$$

whence a complex

$$\begin{aligned} 0 \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \textstyle\bigwedge^n(\mathcal{E}_-) \otimes_{\mathcal{O}_\Gamma} \mathcal{M}_{j-1,l} \longrightarrow \cdots \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{E}_- \otimes_{\mathcal{O}_\Gamma} \mathcal{M}_{j-1,l} \\ \longrightarrow \mathcal{J}_0 \otimes_{\mathcal{O}_\Gamma} \mathcal{M}_{j-1,l} \longrightarrow 0 \end{aligned}$$

According to (i), $\mathcal{M}_{j-1,l}$ is a locally free module. Hence this complex is acyclic.

(iii) Let prove the assertion by induction on j. According to Corollary 8.11, it is true for $j = 0$. Let suppose that it is true for $j-1$. According to the induction hypothesis, for all positive integer i and for $p = 1, \ldots, n$,

$$\mathrm{H}^{i+j-1+p+|l|}(\Gamma, \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{M}_{j-1,(p,l)}) = \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathrm{H}^{i+j-1+p+|l|}(\Gamma, \mathcal{M}_{j-1,(p,l)}) = 0$$

Then, according to (ii), $\mathrm{H}^{i+j+|l|}(\Gamma, \mathcal{M}_{j,l}) = 0$ for all positive integer i since $\mathrm{H}^\bullet$ is an exact δ-functor. □

Proposition 8.14. *The variety $\mathfrak{X}_{0,k}$ has rational singularities and its ideal of definition in $\mathcal{O}_{\Gamma\times\mathfrak{b}^k}$ is the space of global sections of $\mathcal{J}$.*

Proof. From the short exact sequence,

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{\Gamma\times\mathfrak{b}^k} \longrightarrow \mathcal{O}_{E^{(k)}} \longrightarrow 0$$

one deduces the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(\Gamma\times\mathfrak{b}^k, \mathcal{J}) \longrightarrow \mathrm{S}(\mathfrak{b}_-)^{\otimes k} \otimes_{\Bbbk} \mathrm{H}^i(\Gamma, \mathcal{O}_\Gamma) \longrightarrow \mathrm{H}^i(E^{(k)}, \mathcal{O}_{E^{(k)}}) \longrightarrow \mathrm{H}^{i+1}(\Gamma\times\mathfrak{b}^k, \mathcal{J}) \longrightarrow \cdots$$

According to Lemma 8.8, $\mathrm{H}^i(\Gamma, \mathcal{O}_\Gamma) = 0$ for $i > 0$ and according to Lemma 8.12,(iii) and Lemma 8.13,(iii), $\mathrm{H}^i(\Gamma\times\mathfrak{b}^k, \mathcal{J}) = 0$ for $i > 0$. Hence, $\mathrm{H}^i(E^{(k)}, \mathcal{O}_{E^{(k)}}) = 0$ for $i > 0$, whence the short exact sequence

$$0 \longrightarrow \mathrm{H}^0(\Gamma\times\mathfrak{b}^k, \mathcal{J}) \longrightarrow \mathrm{S}(\mathfrak{b}_-)^{\otimes k} \longrightarrow \mathrm{H}^0(E^{(k)}, \mathcal{O}_{E^{(k)}}) \longrightarrow 0$$

Since the image of $\mathrm{S}(\mathfrak{b}_-)^{\otimes k}$ is contained in $\Bbbk[\mathfrak{X}_{0,k}]$, $\Bbbk[\mathfrak{X}_{0,k}] = \Bbbk[\widetilde{\mathfrak{X}_{0,k}}]$ by Corollary 7.4, whence the proposition by Theorem 8.7 since $E^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$ by Lemma 7.2,(iii). □

Corollary 8.15. (i) *The normalization morphism of $\mathcal{C}_{\mathrm{n}}^{(k)}$ is a homeomorphism.*

(ii) *The normalization morphism of $\mathcal{C}^{(k)}$ is a homeomorphism.*

Proof. (i) According to Proposition 3.10, one has the commutative diagram

$$\begin{tikzcd} G\times_B \mathfrak{X}_{0,k} \arrow[r, hook] \arrow[d] & G\times_B \mathfrak{b}^k \arrow[d, "\gamma_{\mathrm{n}}"] \\ \mathcal{C}_{\mathrm{n}}^{(k)} \arrow[r, hook] & \mathcal{B}_{\mathrm{n}}^{(k)} \end{tikzcd}$$

Since $\mathcal{B}_{\mathrm{n}}^{(k)}$ is a normal variety and since $G\times_B \mathfrak{b}^k$ is a desingularization of $\mathcal{B}^{(k)}$ and $\mathcal{B}_{\mathrm{n}}^{(k)}$, the fibers of γ_{n} are connected by Zariski Main Theorem [Mu88, /S 9]. Then the fibers of the restriction of γ_{n} to $G\times_B \mathfrak{X}_{0,k}$ are too since $G\times_B \mathfrak{X}_{0,k}$ is the inverse image of $\mathcal{C}_{\mathrm{n}}^{(k)}$. According to Proposition 8.14, $G\times_B \mathfrak{X}_{0,k}$ is a normal variety. Moreover, the restriction of γ_{n} to $G\times_B \mathfrak{X}_{0,k}$ is projective and birational, whence the commutative diagram

$$\begin{tikzcd} G\times_B \mathfrak{X}_{0,k} \arrow[rr, "\widetilde{\gamma_{\mathrm{n}}}"] \arrow[dr, "\gamma_{\mathrm{n}}"'] & & \widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}} \arrow[dl, "\mu"] \\ & \mathcal{C}_{\mathrm{n}}^{(k)} & \end{tikzcd}$$

with μ the normalization morphism. For x in $\mathcal{C}_{\mathrm{n}}^{(k)}$, $\mu^{-1}(x) = \widetilde{\gamma_{\mathrm{n}}}(\gamma_{\mathrm{n}}^{-1}(x))$. Hence μ is injective since the fibers of γ_{n} are connected, whence the assertion since μ is closed as a finite morphism.

(ii) One has a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}} \arrow[r, "\mu"] \arrow[d, "\widetilde{\eta}"'] & \mathcal{C}_{\mathrm{n}}^{(k)} \arrow[d, "\eta"] \\ \widetilde{\mathcal{C}^{(k)}} \arrow[r, "\mu_0"'] & \mathcal{C}^{(k)} \end{tikzcd}$$

with μ_0 the normalization morphism. According to Proposition 6.8, all fiber of η or $\widetilde{\eta}$ is one single $W(\mathcal{R})$-orbit and by (i), μ is bijective. Hence μ_0 is bijective, whence the assertion since μ_0 is closed as a finite morphism. □

8.6. In this subsection $k = 2$. The open subset E_2 of $E^{(2)}$ identifies with an open subset of $E_{\mathrm{n}}^{(2)}$ and it is B-invariant so that $G\times_B E_2$ is an open subset of $G\times_B E^{(2)}$ and $G\times_B E_{\mathrm{n}}^{(2)}$.

Lemma 8.16. (i) *The variety $G\times_B \mathfrak{X}_{0,2}$ has rational singularities.*

(ii) *The set $G\times_B V_{2,1}$ is a smooth big open subset of $G\times_B \mathfrak{X}_{0,2}$.*

(iii) *The set $G.\iota_{\mathrm{n}}(V_{2,1})$ is a smooth big open subset of $\mathcal{C}_{\mathrm{n}}^{(2)}$.*

(iv) *A global section of $\Omega_{G.\iota_{\mathrm{n}}(V_{2,1})}$ has a regular extension to the smooth locus of $G\times_B \mathfrak{X}_{0,2}$.*

Proof. (i) According to Proposition 8.14, $\mathfrak{X}_{0,2}$ has rational singularities, whence the assertion since $G\times_B \mathfrak{X}_{0,2}$ is a fiber bundle over the the smooth variety G/B with fibers isomorphic to $\mathfrak{X}_{0,2}$.

(ii) According to Proposition 8.3,(iii), $V_{2,1}$ is a smooth big open subset of $\mathfrak{X}_{0,k}$. Then $G\times_B V_{2,1}$ is a smooth big open subset of $G\times_B \mathfrak{X}_{0,2}$ since G/B is smooth.

(iii) Since $\gamma_{\mathrm{n}}^{-1}(G.\iota_{\mathrm{n}}(V_{2,1}))$ equals $G\times_B V_{2,1}$ and since γ_{n} is projective and birational, $G.\iota_{\mathrm{n}}(V_{2,1})$ is a big open subset of $\mathcal{C}_{\mathrm{n}}^{(2)}$. Moreover, $G\times_B V_{2,1}$ is contained in the open subset $\gamma_{\mathrm{n}}^{-1}(W_2)$ of $G\times_B \mathfrak{b}^2$ and the restriction of $\gamma_{\rm n}$ to $\gamma_{\rm n}^{-1}(W_2)$ is an isomorphism onto W_2 by Proposition 3.10,(iv) so that the restriction of $\gamma_{\rm n}$ to $G\times_B V_{2,1}$ is an isomorphism onto $G.\iota_{\rm n}(V_{2,1})$, whence the assertion.

(iv) The assertion results from (iii) and Lemma C.1,(v). □

Corollary 8.17. *The varieties* $\widetilde{\mathcal{C}_{\rm n}^{(2)}}$ *and* $\widetilde{\mathcal{C}^{(2)}}$ *have rational singularities.*

Proof. According to the proof of Corollary 8.15, one has the following commutative diagram:

$$\begin{tikzcd} G\times_B \mathfrak{X}_{0,2} \arrow[rr, "\widetilde{\gamma_{\rm n}}"] \arrow[dr, "\gamma_{\rm n}"'] & & \widetilde{\mathcal{C}_{\rm n}^{(2)}} \arrow[dl, "\mu"] \\ & \mathcal{C}_{\rm n}^{(2)} & \end{tikzcd}$$

with μ the normalization morphism. Moreover, $\widetilde{\gamma_{\rm n}}$ is a projective and birational morphism. By Lemma 8.16,(iii), $\mu^{-1}(G.\iota_{\rm n}(V_{2,1}))$ is a smooth big open subset of $\widetilde{\mathcal{C}_{\rm n}^{(2)}}$ and the restriction of μ to $\mu^{-1}(G.\iota_{\rm n}(V_{2,1}))$ is an isomorphism onto $G.\iota_{\rm n}(V_{2,1})$. So, by Lemma 8.16,(iv), all global section of $\Omega_{\mu^{-1}(G.\iota_{\rm n}(V_{2,1}))}$ has a regular extension to the smooth locus of $G\times_B \mathfrak{X}_{0,2}$. According to Proposition 7.3,(ii), $G\times_B E^{(2)}$ is a desingularization of $\mathcal{C}_{\rm n}^{(2)}$ and $E^{(2)}$ is a desingularization of $\mathfrak{X}_{0,2}$ with a B-equivariant desingularization morphism by Lemma 7.2,(ii). Hence $G\times_B E^{(2)}$ is a desingularization of $\widetilde{\mathcal{C}_{\rm n}^{(2)}}$ and $G\times_B \mathfrak{X}_{0,2}$. As a result by Lemma 8.16,(i) and [KK73, p.50], all global section of $\Omega_{\mu^{-1}(G.\iota_{\rm n}(V_{2,1}))}$ has a regular extension to $G\times_B E^{(2)}$. According to Proposition 6.6, $\widetilde{\mathcal{C}_{\rm n}^{(2)}}$ is the normalization of the isospectral commuting variety and according to [Gi11, Theorem 1.3.4], $\widetilde{C_{\rm n}^{(2)}}$ is Gorenstein. Hence by [KK73, p.50], $\widetilde{\mathcal{C}_{\rm n}^{(2)}}$ has rational singularities. By Proposition 6.8,(ii), $\widetilde{\mathcal{C}^{(2)}}$ is the categorical quotient of $\widetilde{\mathcal{C}_{\rm n}^{(2)}}$ under the action of $W(\mathcal{R})$. So, by [El81, Lemme 1], $\widetilde{\mathcal{C}^{(2)}}$ has rational singularities. □

Appendix A. Notations.

In this appendix, V is a finite dimensional vector space. Let denote by $\mathrm{S}(V)$ and $\bigwedge(V)$ the symmetric and exterior algebras of V respectively. For all integer i, $\mathrm{S}^i(V)$ and $\bigwedge^i(V)$ are the subspaces of degree i for the usual gradation of $\mathrm{S}(V)$ and $\bigwedge(V)$ respectively. In particular, $\mathrm{S}^i(V)$ and $\bigwedge^i(V)$ are equal to zero for i negative.

- For l positive integer, let denote by $\mathfrak{S}_l$ the group of permutations of l elements.
- For m positive integer and for $l=(l_1,\ldots,l_m)$ in $\mathbb{N}^m$, let set:

$$\begin{aligned} |l| &:= l_1+\cdots+l_m \\ \mathrm{S}^l(V) &:= \mathrm{S}^{l_1}(V)\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}\mathrm{S}^{l_m}(V) \\ \textstyle\bigwedge^l(V) &:= \textstyle\bigwedge^{l_1}(V)\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}\bigwedge^{l_m}(V) \end{aligned}$$

- For k positive integer and for $l=(l_1,\ldots,l_m)$ in $\mathbb{N}^m$ such that $1\leq |l|\leq k$, let denote by $V^{\otimes k}$ the k-th tensor power of V and let denote by $\mathfrak{S}_l$ the direct product $\mathfrak{S}_{l_1}\times\cdots\times\mathfrak{S}_{l_m}$. The group $\mathfrak{S}_l$ has a natural action on $V^{\otimes k}$ given by

$$(\sigma_1,\dots,\sigma_m).(v_1\otimes\cdots\otimes v_k) = \begin{array}{l} v_{\sigma_1(1)}\otimes\cdots\otimes v_{\sigma_1(l_1)}\otimes v_{l_1+\sigma_2(1)}\otimes\cdots\otimes v_{l_1+\sigma_2(l_2)} \\ \otimes\cdots\otimes v_{|l|-l_m+\sigma_m(1)}\otimes\cdots\otimes v_{|l|-l_m+\sigma_m(l_m)}\otimes v_{|l|+1}\otimes\cdots\otimes v_k\end{array}$$

The map

$$a \longmapsto \pi_{k,l}(a) := \prod_{j=1}^{m} \frac{1}{l_j!}\sum_{\sigma\in\mathfrak{S}_l} \sigma.a$$

is a projection of $V^{\otimes k}$ onto $(V^{\otimes k})^{\mathfrak{S}_l}$. Moreover, the restriction to $(V^{\otimes k})^{\mathfrak{S}_l}$ of the canonical map from $V^{\otimes k}$ to $\mathrm{S}^l(V)\otimes_{\Bbbk} V^{\otimes(k-|l|)}$ is an isomorphism of vector spaces.

Appendix B. Some complexes.

Let X be a smooth algebraic variety. For $\mathcal{M}$ a coherent $\mathcal{O}_X$-module and for k positive integer, let denote by $\mathcal{M}^{\otimes k}$ the k-th tensor power of $\mathcal{M}$. According to Notations A, for all l in $\mathbb{N}^m$ such that $|l|\leq k$, there is an action of $\mathfrak{S}_l$ on $\mathcal{M}^{\otimes k}$. Moreover, $\mathrm{S}^l(\mathcal{M})$ and $\bigwedge^l(\mathcal{M})$ are coherent modules defined by the same formulas as in Notations A.

B.1. Let $D(V)$ be the algebra $\mathrm{S}(V)\otimes_{\Bbbk}\bigwedge(V)$ and let d be the $\bigwedge(V)$-derivation of $D(V)$ such that $\mathrm{d}v{\otimes}a = 1{\otimes}(v\wedge a)$ for all (v,a) in $V\times\bigwedge(V)$. The gradation of $\bigwedge(V)$ induces on $D(V)$ a gradation so that $D(V)$ is a graded cohomology complex denoted by $D^{\bullet}(V)$. For k positive integer, let denote by $D^{\bullet}_k(V)$ the graded subcomplex of $D^{\bullet}(V)$ whose space of degree i is $\mathrm{S}^{k-i}(V)\otimes_{\Bbbk}\bigwedge^i(V)$:

$$D^{\bullet}_k(V) := \bigoplus_{i=0}^{k} D^i_k(V) = \bigoplus_{i=0}^{k}\mathrm{S}^{k-i}(V)\otimes_{\Bbbk}\textstyle\bigwedge^i(V)$$

Lemma B.1. *Let k be a positive integer.*
(i) *The cohomology of $D^{\bullet}(V)$ equals $\Bbbk$.*
(ii) *For k positive, the subcomplex $D^{\bullet}_k(V)$ of $D^{\bullet}(V)$ is acyclic.*

Proof. (i) We prove the assertion by induction on $\dim V$. Let denote by d the differential of $D^{\bullet}(V)$. The cohomology in degree 0 of $D^{\bullet}(V)$ equals $\Bbbk$. For $\dim V = 1$, $D^{\bullet}(V)$ has no cohomology in positive degree since $\mathrm{d}v^m{\otimes}1 = mv^{m-1}{\otimes}v$ for all v in V. Let suppose that it is true for all vector space of dimension at most $\dim V-1$. Let a be an homogeneous cocycle of positive degree d, let W be a subspace of codimension 1 of V and let v be in $V\setminus W$. Then a has a unique expansion

$$a = v^m(a'_m + a''_m\wedge v)+\cdots+a'_0+a''_0\wedge v,$$

with a'_i and a''_i in $D^d(W)$ and $D^{d-1}(W)$ respectively for $i=0,\dots,m$. From the equality

$$\mathrm{d}a = \sum_{i=0}^{m} v^i(\mathrm{d}a'_i + (\mathrm{d}a''_i)\wedge v) + \sum_{i=1}^{m}(-1)^d iv^{i-1}a'_i\wedge v$$

one deduces that a'_m and a''_m are cocycles of degree d and $d-1$ respectively of $D^{\bullet}(V)$ since a is a cocycle. Hence by induction hypothesis, $a'_m = \mathrm{d}b'_m$ for some element b'_m of $D^{d-1}(W)$. If $d>1$, by induction hypothesis again, $a''_m = \mathrm{d}b''_m$ for some element b''_m of $D^{d-2}(W)$. As a result,

$$a - \mathrm{d}v^m(b'_m + b''_m \wedge v) = (-1)^d m v^{m-1} b'_m \wedge v + \sum_{i=0}^{m-1} v^i(a'_i + a''_i \wedge v).$$

So by induction on m, a is a coboundary. Let suppose $d = 1$. Since a''_m is a cocycle, it is in $\Bbbk$. Then

$$a - \mathrm{d}(v^m b'_m + \frac{1}{m+1} a''_m v^{m+1}) = -m v^{m-1} b'_m \wedge v + \sum_{i=0}^{m-1} v^i(a'_i + a''_i \wedge v).$$

So by induction on m, a is a coboundary, whence the assertion.

(ii) Since $D^\bullet(V)$ is the direct sum of the subcomplexes $D^\bullet_k(V)$, $k \in \mathbb{N}$, the assertion results from (i). □

B.2. Let $\mathcal{E}$ and $\mathcal{M}$ be locally free $\mathcal{O}_X$-modules.

Proposition B.2. *Let i be a positive integer and let suppose that*

$$\mathrm{H}^{i+j}(X, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$$

for all nonnegative integers j, k.

(i) *For all positive integers m and k and for all l in $\mathbb{N}^m$ such that $|l| \leq k$,*

$$\mathrm{H}^i(X, \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$$

(ii) *For all positive integers n_1, n_2, k and for all (l, m) in $\mathbb{N}^{n_1} \times \mathbb{N}^{n_2}$ such that $|l| + |m| \leq k$,*

$$\mathrm{H}^i(X, \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \textstyle\bigwedge^m(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-|m|)} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$$

Proof. (i) Let $\mathcal{U}$ be an affine open cover of X so that the cohomology of the Čech complexes $C^\bullet(\mathcal{U}, \mathcal{E}^{\otimes k})$ and $C^\bullet(\mathcal{U}, \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)})$ are the cohomology of the $\mathcal{O}_X$-modules $\mathcal{E}^{\otimes k}$ and $\mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)}$ respectively. The action of $\mathfrak{S}_l$ on $\mathcal{E}^{\otimes k}$ induces an action on $C^\bullet(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M})$ commuting with its derivation denoted by d. Let $\overline{\varphi}$ be a cocycle of degree i of

$$C^\bullet(\mathcal{U}, \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)} \otimes_{\mathcal{O}_X} \mathcal{M})$$

and let φ be the representative of $\overline{\varphi}$ in $C^j(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M})^{\mathfrak{S}_l}$. Then φ is a cocycle of degree i of $C^\bullet(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M})$. By hypothesis, for some ψ in $C^{i-1}(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M})$, $\varphi = \mathrm{d}\psi$. Then, since φ is invariant under $\mathfrak{S}_l$ and since d commutes with the action of $\mathfrak{S}_l$, $\varphi = \mathrm{d}\psi^\#$. Hence $\overline{\varphi}$ is the coboundary of the image of $\psi^\#$ in $C^{i-1}(\mathcal{U}, \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)} \otimes_{\mathcal{O}_X} \mathcal{M})$, whence the assertion.

(ii) Let suppose $n_2 = 1$ and let prove the assertion by induction on m. Since $\mathcal{E}$ is a locally free module, according to Lemma B.1,(ii), one has a long exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathrm{S}^m(\mathcal{E}) \longrightarrow \mathrm{S}^{m-1}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \cdots \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \textstyle\bigwedge^{m-1}(\mathcal{E}) \longrightarrow \bigwedge^m(\mathcal{E}) \longrightarrow 0$$

whence an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{S}^{l^{(m)}}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathrm{S}^{l^{(m-1)}}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \\ \longrightarrow \mathrm{S}^{l^{(1)}}(\mathcal{E}) \otimes_{\mathcal{O}_X} \textstyle\bigwedge^{m-1}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \\ \longrightarrow \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \textstyle\bigwedge^m(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow 0 \end{aligned}$$

with $l^{(j)} = (l, j)$ in $\mathbb{N}^{n_1+1}$ for all j in $\mathbb{N}$, since $\mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-m)}$ and $\mathcal{M}$ are locally free modules. According to the induction hypothesis for $j > 0$,

$$\mathrm{H}^{i+j-1}(X, \mathrm{S}^{l^{(j)}}(\mathcal{E}) \otimes_{\mathcal{O}_X} \textstyle\bigwedge^{m-j}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-m)} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$$

Then,

$$\mathrm{H}^{i}(X, \mathrm{S}^{l}(\mathcal{E}) \otimes_{\mathcal{O}_X} \textstyle\bigwedge^{m}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-m)} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$$

since $\mathrm{H}^\bullet$ is an exact δ-functor.

Let suppose the assertion true for $n_2 - 1$ and let prove the assertion by induction on m_{n_2}. According to the induction hypothesis, it is true for $m_{n_2} = 0$. According to Lemma B.1,(ii), one has a long exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathrm{S}^{m_{n_2}}(\mathcal{E}) \longrightarrow \mathrm{S}^{m_{n_2}-1}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \cdots \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \textstyle\bigwedge^{m_{n_2}-1}(\mathcal{E}) \longrightarrow \textstyle\bigwedge^{m_{n_2}}(\mathcal{E}) \longrightarrow 0$$

Tensoring this sequence by the locally free module

$$\mathrm{S}^{l}(\mathcal{E}) \otimes_{\mathcal{O}_X} \textstyle\bigwedge^{m'}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-|m|} \otimes_{\mathcal{O}_X} \mathcal{M}$$

with $m' = (m_1, \ldots, m_{n_2-1})$ and arguing as before, we deduce the equality

$$\mathrm{H}^{i}(X, \mathrm{S}^{l}(\mathcal{E}) \otimes_{\mathcal{O}_X} \textstyle\bigwedge^{m}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-|m|} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$$

from the induction hypothesis, whence the assertion. □

B.3. Let W be a subspace of V and let set $E := V/W$. Let $C_\bullet^{(n)}(V, W)$, $n = 1, 2, \ldots$ be the sequence of graded spaces over $\mathbb{N}$ defined by the induction relations:

$$C_0^{(1)}(V, W) := V \qquad C_1^{(1)}(V, W) := W \qquad C_i^{(1)}(V, W) := 0$$

$$C_0^{(n)}(V, W) := V^{\otimes n} \qquad C_j^{(n)}(V, W) := C_j^{(n-1)}(V, W) \otimes_{\mathbb{k}} V \oplus C_{j-1}^{(n-1)}(V, W) \otimes_{\mathbb{k}} W$$

for $i \geq 2$ and $j \geq 1$.

Lemma B.3. *Let n be a positive integer. There exists a graded differential of degree -1 on $C_\bullet^{(n)}(V, W)$ such that the complex so defined has no homology in positive degree.*

Proof. Let prove the lemma by induction on n. For $n = 1$, d is given by the canonical injection of W in V. Let suppose that $C_\bullet^{(n-1)}(V, W)$ has a differential d verifying the conditions of the lemma. For $j > 0$, let denote by δ the linear map

$$C_j^{(n)}(V, W) \longrightarrow C_{j-1}^{(n)}(V, W) \qquad (a{\otimes}v, b{\otimes}w) \longmapsto (\mathrm{d}a{\otimes}v + (-1)^j b{\otimes}w, \mathrm{d}b{\otimes}w)$$

with a, b, v, w in $C_j^{(n-1)}(V, W)$, $C_{j-1}^{(n-1)}(V, W)$, V, W respectively. Then δ is a graded differential of degree -1. Let c be a cycle of positive degree j of $C_\bullet^{(n)}(V, W)$. Then c has an expansion

$$c = (\sum_{i=1}^{d} a_i{\otimes}v_i, \sum_{i=1}^{d'} b_i{\otimes}v_i)$$

with $v_1,\dots,v_d$ a basis of V such that $v_1,\dots,v_{d'}$ is a basis of W and with $a_1,\dots,a_d$ and $b_1,\dots,b_{d'}$ in $C_j^{(n-1)}(V,W)$ and $C_{j-1}^{(n-1)}(V,W)$ respectively. Since c is a cycle,

$$\sum_{i=1}^{d} \mathrm{d}a_i{\otimes}v_i + (-1)^j \sum_{i=1}^{d'} b_i{\otimes}v_i = 0$$

Hence $b_i = (-1)^{j+1}\mathrm{d}a_i$ for $i = 1,\dots,d'$ so that

$$c + \delta(\sum_{i=1}^{d'}(-1)^j a_i{\otimes}v_i, 0) = (\sum_{i=1}^{d} a_i{\otimes}v_i + \sum_{i=1}^{d'} a_i{\otimes}v_i, \sum_{i=1}^{d'}(b_i{\otimes}v_i + (-1)^j\mathrm{d}a_i{\otimes}v_i)) = (\sum_{i=1}^{d} a_i{\otimes}v_i + \sum_{i=1}^{d'} a_i{\otimes}v_i, 0)$$

So one can suppose $b_1,\dots,b_{d'}$ all equal to 0. Then $a_1,\dots,a_d$ are cycles of degree j of $C_\bullet^{(n-1)}(V,W)$. By induction hypothesis, they are boundaries of $C_\bullet^{(n-1)}(V,W)$ so that c is a boundary of $C_\bullet^{(n)}(V,W)$, whence the lemma. □

Remark B.4. The results of this subsection remain true for V or W of infinite dimension since a vector space is an inductive limit of finite dimensional vector spaces.

Appendix C. Rational Singularities.

Let X be an affine irreducible normal variety and let X' be a smooth big open subset of X.

Lemma C.1. *Let Y be an irreducible Gorenstein variety Y and let π be a projective birational morphism from Y to X. Let denote by $\mathcal{K}$ the canonical module of Y. Let suppose that the following conditions are verified:*

(1) *the open subset $\pi^{-1}(X')$ of Y is big,*
(2) *the restriction of π to $\pi^{-1}(X')$ is an isomorphism onto X'.*

Let denote by J the space of global sections of $\mathcal{K}$ and let $\mathcal{J}$ be the localization of J on X.

(i) *The algebra $\Bbbk[X]$ is the space of global sections of $\mathcal{O}_Y$ and Y is a normal variety.*

(ii) *For all open subset O of X and for all local section a of $\mathcal{J}$ over $O\cap X'$, a is the restriction to $O\cap X'$ of one and only one local section of $\mathcal{J}$ over O.*

(iii) *The $\mathcal{O}_Y$-modules $\pi^*(\mathcal{J})$ and $\mathcal{K}$ are equal.*

(iv) *For all injective $\Bbbk[X]$-module I, the canonical morphism*

$$J \otimes_{\Bbbk[X]} \mathrm{Hom}_{\Bbbk[X]}(J, I) \longrightarrow I$$

is an isomorphism.

(v) *All regular form of top degree on X' has a unique regular extension to the smooth locus of Y.*

Proof. (i) If $Y' \to Y$ is a desingularization of Y, $Y' \to X$ is a desingularization of X since π is projective and birational. Moreover, all global section of $\mathcal{O}_Y$ is a global section of $\mathcal{O}_{Y'}$, whence, by Lemma 1.1, $\Bbbk[X]$ is the space of global sections of $\mathcal{O}_Y$ since X is normal. According to Conditions (1) and (2), $\pi^{-1}(X')$ is a smooth big open subset of Y. So, by Serre's normality criterion [Bou98, §1, no 10, Théorème 4], Y is normal since Y is Gorenstein.

(ii) Since $\mathcal{J}$ is the localization of J on X, it suffices to prove the assertion for $O = X$. Let a be a local section of $\mathcal{J}$ over X'. According to (2), $\pi^*(a)$ is a local section of $\mathcal{K}$ over $\pi^{-1}(X')$. Since Y is Gorenstein,

$\mathcal{K}$ is locally free of rank 1. So, there is an affine open cover $V_1,\dots,V_l$ of Y such that the restriction of $\mathcal{K}$ to V_i is a free $\mathcal{O}_{V_i}$-module of rank 1. Let p_i be a generator of this module. Setting $V'_i := V_i \cap \pi^{-1}(X')$, for some regular function a'_i on V'_i, $a'_i p_i$ is the restriction of $\pi^*(a)$ to V'_i. According to (i), a'_i has a regular extension to V_i since V'_i is a big open subset of V_i by Condition (1). Let denote by a_i this extension. Then, for $1 \leq i, j \leq l$, the restrictions of $a_i p_i$ and $a_j p_j$ to $V_i \cap V_j$ are two local sections of $\mathcal{K}$ over $V_i \cap V_j$ which are equal on $V'_i \cap V'_j$. Hence $a_i p_i$ and $a_j p_j$ have the same restriction to $V_i \cap V_j$ since $\mathcal{K}$ is torsion free as a locally free module. As a result, $\pi^*(a)$ is the restriction to $\pi^{-1}(X')$ of a unique global section of $\mathcal{K}$ since $\mathcal{K}$ is torsion free.

(iii) Let a be in $\Bbbk[V_i]\otimes_{\Bbbk[X]} J$. By condition (2), for some regular function a' on V'_i, $a' p_i$ is the restriction of a to V'_i. Since V_i is normal and since V'_i is a big open subset of V_i, a' has a regular extension to V_i so that a is in $\Gamma(V_i, \mathcal{K})$. Conversely, let a be in $\Gamma(V_i, \mathcal{K})$. Since V'_i is a big open subset of V_i, for some open subset V''_i of X, V_i is contained in $\pi^{-1}(V''_i)$ and $X' \cap V''_i$ equals $\pi(V'_i)$. By Condition (2), for some a' in $\Gamma(\pi(V'_i), \mathcal{J})$, $\pi^*(a')$ is the restriction of a to V'_i. According to (ii), a' is the restriction to $\pi(V'_i)$ of a unique local section a'' of $\mathcal{J}$ over V''_i. Then the restriction of $\pi^*(a'')$ to V_i equals a since a and $\pi^*(a'')$ have the same restriction to V'_i and since $\mathcal{K}$ is torsion free, whence the assertion.

(iv) Let denote by ψ the canonical morphism

$$J \otimes_{\Bbbk[X]} \mathrm{Hom}_{\Bbbk[X]}(J, I) \longrightarrow I \qquad a \otimes \varphi \longmapsto \varphi(a)$$

Let x be in I and let a be in $J \setminus \{0\}$. Since I is an injective module, I is divisible so that $x = bx'$ for some x' in I. Denoting by φ the morphism $c \mapsto cx'$ from J to I, $\psi(b \otimes \varphi) = x$. So, ψ is surjective.

Let denote by K the kernel of ψ and let suppose K different from 0. One expects a contradiction. Let φ be in K. For $i = 1,\dots,l$, let set:

$$J_i := \Bbbk[V_i] \otimes_{\Bbbk[X]} J \qquad I_i := \Bbbk[V_i] \otimes_{\Bbbk[X]} I$$

so that

$$\Bbbk[V_i] \otimes_{\Bbbk[X]} J \otimes_{\Bbbk[X]} \mathrm{Hom}_{\Bbbk[X]}(J, I) = J_i \otimes_{\Bbbk[V_i]} \mathrm{Hom}_{\Bbbk[V_i]}(J_i, I_i)$$

and let denote by ψ_i the canonical morphism

$$J_i \otimes_{\Bbbk[V_i]} \mathrm{Hom}_{\Bbbk[V_i]}(J_i, I_i) \longrightarrow I_i$$

so that the restriction of $\pi^*(\varphi)$ to V_i is in the kernel of ψ_i. According to (iii), J_i is the free $\Bbbk[V_i]$-module generated by p_i so that the morphism

$$I_i \longrightarrow J_i \otimes_{\Bbbk[V_i]} \mathrm{Hom}_{\Bbbk[V_i]}(J_i, I_i) \qquad x \longmapsto p_i \otimes \varphi_x \qquad \text{with} \qquad \varphi_x(ap_i) = ax$$

is an isomorphism equals to the inverse of ψ_i. Hence the restriction of $\pi^*(\varphi)$ to V_i equals 0. As a result, $\pi^*(\varphi) = 0$. Hence $\pi^*(\mathcal{O}_X \otimes_{\Bbbk[X]} K) = 0$. Since K is different from 0, K contains a finitely generated submodule K', different from 0. Then, for some locally closed subvariety $X_{K'}$ of X, $\mathcal{O}_{X_{K'}} \otimes_{\Bbbk[X]} K'$ is a free $\mathcal{O}_{X_{K'}}$-module different from 0. Denoting, by $\pi_{K'}$ the restriction of π to $\pi^{-1}(X_{K'})$, $\pi^*_{K'}(\mathcal{O}_{X_{K'}} \otimes_{\Bbbk[X]} K')$ is different from zero, whence the contradiction since it is the restriction to $\pi_{K'}^{-1}(X_{K'})$ of $\pi^*(\mathcal{O}_X \otimes_{\Bbbk[X]} K')$.

(v) Let Y' be the smooth locus of Y. According to Condition (2), $\pi^{-1}(X')$ is a dense open subset of Y'. Moreover, $\pi^{-1}(X')$ identifies with X'. Let ω be a differential form of top degree on X'. Since $\Omega_{Y'}$ is a locally free module of rank one, there is an affine open cover $O_1,\dots,O_k$ on Y' such that restriction of $\Omega_{Y'}$ to O_i is a free $\mathcal{O}_{O_i}$-module generated by some section ω_i. For $i = 1,\dots,k$, let set $O'_i := O_i \cap X'$. Let

ω be a regular form of top degree on X'. For $i = 1, \ldots, k$, for some regular function a_i on O'_i, $a_i\omega_i$ is the restriction of ω to O'_i. According to Condition (1), O'_i is a big open subset of O_i. Hence a_i has a regular extension to O_i since O_i is normal. Denoting again by a_i this extension, for $1 \leq i, j \leq k$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O'_i \cap O'_j$ and $O_i \cap O_j$ since $\Omega_{Y'}$ is torsion free as a locally free module. Let ω' be the global section of $\Omega_{Y'}$ extending the $a_i\omega_i$'s. Then ω' is a regular extension of ω to Y' and this extension is unique since X' is dense in Y' and since $\Omega_{Y'}$ is torsion free. □

Proposition C.2. *Let suppose that there exist an irreducible Gorenstein variety Y, with rational singularities, and a projective birational morphism π from Y to X verifying Conditions* (1) *and* (2) *of Lemma* C.1. *Then X has rational singularities.*

Proof. Let Y' be the smooth locus of Y. According to [Hir64], there exists a desingularization Z of Y, with morphism τ, such that the restriction of τ to $\tau^{-1}(Y')$ is an isomorphism onto Y'. According to Lemma C.1,(v), all regular differential form of top degree on the smooth locus of X has a regular extension to Y'. Since Y has rational singularities and since Z is a desingularization of Y, all regular differential form of top degree on the smooth locus of Y has a regular extension to Z by [KK73, p.50]. Hence all regular differential form of top degree on the smooth locus of X has a regular extension to Z. Since Z is a desingularization of Y and since π is projective and birational, Z is a desingularization of X. So, by [KK73, p.50] again, it remains to prove that X is Cohen-Macaulay.

Since Z, Y, X are varieties over $\Bbbk$, one a has the commutative digrams

$$\begin{tikzcd} Z \arrow[rr, "\tau"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \mathrm{Spec}(\Bbbk) & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "\pi"] \arrow[dr, "q"'] & & X \arrow[dl, "r"] \\ & \mathrm{Spec}(\Bbbk) & \end{tikzcd}$$

According to [Hi91, 4.3,(iv)], $p^!(\Bbbk)$, $q^!(\Bbbk)$, $r^!(\Bbbk)$ are dualizing complexes over Z, Y, X respectively. Furthermore, by [Hi91, 4.3,(ii)], $p^!(\Bbbk)[-\dim Z]$ equals Ω_Z and since Y is Gorenstein, the cohomology of $q^!(\Bbbk)[-\dim Z]$ is concentrated in degree 0 and equals the canonical module $\mathcal{K}$ of Y. Let set $\mathcal{D} := r^!(\Bbbk)[-\dim Z]$ so that $\pi^!(\mathcal{D}) = \mathcal{K}$ and $(\pi\circ\tau)^!(\mathcal{D}) = \Omega_Z$ by [Hi91, 4.3,(iv)]. Since τ and π are projective morphisms, one has the isomorphisms

$$\mathrm{R}(\tau)_*(\mathrm{R}\mathscr{H}om_Z(\Omega_Z, \Omega_Z)) \longrightarrow \mathrm{R}\mathscr{H}om_Y(\mathrm{R}(\tau)_*(\Omega_Z), \mathcal{K})$$

$$\mathrm{R}(\pi)_*(\mathrm{R}\mathscr{H}om_Y(\mathcal{K}, \mathcal{K})) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}(\tau)_*(\mathcal{K}), \mathcal{D})$$

by [Hi91, 4.3,(iii)]. Since Ω_Z and $\mathcal{K}$ are locally free of rank 1,

$$\mathrm{H}^i(\mathrm{R}\mathscr{H}om_Z(\Omega_Z, \Omega_Z)) = \begin{cases} \mathcal{O}_Z & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i > 0 \end{cases}$$

$$\mathrm{H}^i(\mathrm{R}\mathscr{H}om_Y(\mathcal{K}, \mathcal{K})) = \begin{cases} \mathcal{O}_Y & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i > 0 \end{cases}$$

the left hand sides can be identified to $\mathrm{R}(\tau)_*(\mathcal{O}_Z)$ and $\mathrm{R}(\pi)_*(\mathcal{K})$ respectively, whence an isomorphism

$$\mathrm{R}(\pi)_*(\mathcal{O}_Y) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}(\pi)_*(\mathcal{K}), \mathcal{D})$$

Let J be the space of global sections of $\mathcal{K}$. According to Grauert-Riemenschneider Theorem [GR70], denoting by $\mathcal{J}$ the localization of J on X,

$$\mathrm{R}(\tau)_*(\Omega_Z) = \mathcal{K} \qquad \mathrm{R}(\pi\circ\tau)_*(\Omega_Z) = \mathcal{J}$$

whence $\mathrm{R}(\pi)_*(\mathcal{K}) = \mathcal{J}$ and one has an isomorphism

$$\mathrm{R}(\pi)_*(\mathcal{O}_Y) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathcal{J}, \mathcal{D})$$

According to Lemma C.1,(iv), there is an isomorphism

$$\mathcal{J} \otimes^{\mathrm{L}} \mathrm{R}\mathscr{H}om_X(\mathcal{J}, \mathcal{D}) \longrightarrow \mathcal{D}$$

in the derived category $\mathrm{D}^+(X)$ of complexes bounded below of $\mathcal{O}_X$-modules, whence an isomorphism

$$\mathcal{J} \otimes^{\mathrm{L}} \mathrm{R}(\pi)_*(\mathcal{O}_Y) \longrightarrow \mathcal{D}$$

According to Lemma C.1,(iii), $\pi^*(\mathcal{J}) = \mathcal{K}$. Then, since $\mathcal{J} = \mathrm{R}(\pi)_*(\mathcal{K})$, one has an isomorphism

$$\mathcal{J} \otimes^{\mathrm{L}} \mathrm{R}(\pi)_*(\mathcal{O}_Y) \longrightarrow \mathrm{R}(\pi)_*(\mathcal{K} \otimes \mathcal{O}_Y)$$

by the projection formula [Mebk89, Appendice B]. So, since the right hand side equals $\mathcal{J}$, there is an isomorphism

$$\mathcal{J} \longrightarrow \mathcal{D}$$

in $\mathrm{D}^+(X)$. As a result, the cohomology of the dualizing complex $\mathcal{D}$ of X is concentrated in degree 0. Hence X is Cohen-Macaulay [El78]. □

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